A note on a generalized Black-Scholes formula

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Consider the following model of asset pricing

$$\frac{dS}{S} = mdt + dY_t \tag{1}$$

where S_t is a stock price at time t, m is a known constant, process Y_t is defined by the following stochastic differential equation

$$dY_t = -bY_t dt + \sigma dW_t, \quad b \geq 0, \quad \sigma > 0, \quad Y_0 = 0. \tag{2}$$

Here W_t is a standard Brownian motion under the true probability measure P. Observe that process Y_t is a mean reverting Ornstein-Uhlenbeck process for $b > 0$, and it is a Wiener process with zero mean and standard deviation σ for $b = 0$. Therefore in the case of $b = 0$ (1) becomes the Black-Scholes model of asset pricing. We also assume that the risk free interest rate r is a constant.
Below we prove the following result.

Theorem
Assume (1) and (2) and denote by $C(S_t, t)$ the value of a standard European call option written on the stock S_t which does not pay the dividends. K is a strike price and T is the expiration date. Then the

price of the call option at time $t = 0$, $\quad C_0 = C(S_0, 0)$, can be expressed as

$$C_0 = S_0 exp\{(s_b^2 - \sigma^2 T)/2\}\Phi(\frac{log(S_0/K) + (r + \sigma^2/2)T + (s_b^2 - \sigma^2 T)}{s_b})$$

$$-Ke^{-rT}\Phi(\frac{log(S_0/K) + (r - \sigma^2/2)T}{s_b}), \quad b > 0 \qquad (3)$$

where $s_b^2 = \frac{\sigma^2}{2b}\{1 - exp(-2bT)\}$, $\quad$ and $\quad \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} exp(-z^2/2)dz.$

Note: To see that formula (3) is a generalization of the Black Scholes formula presented in Black and Scholes(1973) note the following

$$s_b^2 - \sigma^2 T = -\sigma^2 T^2 b(1 + o(1)), \quad b \to 0.$$

Consequently $s_b \to \sigma\sqrt{T}$ $\quad$ and $\quad s_b^2 - \sigma^2 T \to 0$ when $\quad b \to 0$. Therefore, when b converges to zero the RHS of (3) converges to

$$S_0\Phi(\frac{log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}) - Ke^{-rT}\Phi(\frac{log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}).$$

Proof

Using Ito's lemma one can find the following solution to (1)

$$S(t) = S_0 exp\{(m - \frac{1}{2}\sigma^2)t + Y_t\} \qquad (4)$$

The solution to (2) has the following integral representation

$$Y_t = \sigma \int_0^t exp\{-b(t - s)\}dW_t$$

(see, for example, Steele(2000), chapter 9).

Now we define the risk neutral probability measure Q. Let Q be chosen so that the process W_t^Q defined by

$$dW_t^Q = dW_t + \frac{m-r}{\sigma}(bt+1)dt, \tag{5}$$

is a Q- Brownian motion.

Define the following process

$$Z_t = \sigma \int_0^t exp\{-b(t-s)\}dW_t^Q, \qquad \text{with} \quad Z_0 = 0. \tag{6}$$

Due to its definition and Ito's lemma Z_t satisfies the following stochastic differential equation

$$dZ_t = -bZ_t dt + \sigma dW_t^Q,$$

From (5) and (6) we have

$$\begin{aligned} Z_t &= \sigma \int_0^t exp\{-b(t-s)\}(dW_s + \frac{m-r}{\sigma}(bs+1)ds) \\ &= Y_t + (m-r)\int_0^t exp\{-b(t-s)\}(bs+1)ds. \end{aligned}$$

Now it is easy to see that

$$Z_t = (m-r)t + Y_t \tag{7}$$

From (4) and (7) we find that

$$S(t) = S_0 exp\{(r - \frac{1}{2}\sigma^2)t + Z_t\} \tag{8}$$

The last representation shows that S_t is a solution to the following stochastic differential equation

$$\frac{dS}{S} = rdt + dZ_t$$

Now we turn to the computation of C_0. Due to the Feynman-Kac formula

$$C_0 = e^{-rT} E_Q[(S_T - K)_+]. \tag{9}$$

Since

$$Var_Q(Z_t) = E_Q Z_t^2 = \sigma^2 \int_0^t exp\{-2b(t-s)\}ds = \frac{\sigma^2}{2b}[1-exp(-2bt)] = s_b^2,$$

we note that random variable Z_t is distributed normally with zero mean and variance s_b^2. Therefore random variable $\xi = Z_T - \frac{1}{2}\sigma^2 T$ is also normally distributed with mean $-\frac{1}{2}\sigma^2$ and variance s_b^2 under Q measure. Using above findings and (8) we can rewrite (9) as

$$\begin{aligned} e^{rT}C_0 &= E_Q[(S_0 e^{rT+\xi} - K)_+] \\ &= \frac{1}{\sqrt{2\pi}s_b}[S_0 e^{rT} \int_{log(K/S_0)-rT}^{\infty} exp\{y - (y+\sigma^2 T/2)^2/(2s_b^2)\}dy \\ &\qquad -K \int_{log(K/S_0)-rT}^{\infty} exp\{-(y+\sigma^2 T/2)^2/(2s_b^2)\}dy]. \end{aligned}$$

Therefore

$$e^{rT}C_0 = \frac{1}{\sqrt{2\pi}s_b}[S_0 e^{rt} I_1 - K I_2], \tag{10}$$

where

$$\begin{aligned} I_1 &= \int_A^{\infty} exp\{y - (y+\alpha)^2/(2s_b^2)\}dy \\ I_2 &= \int_A^{\infty} exp\{-(y+\alpha)^2/(2s_b^2)\}dy \end{aligned}$$

with

$$A = log(K/S_0)-rT \quad \text{and} \quad \alpha = \sigma^2 T/2. \tag{11}$$

After some algebraic rearrangements it is easy to see that

$$I_1 = (\sqrt{2\pi}s_b)exp\{s_b^2/2-\alpha\}\Phi(\frac{s_b^2 - A - \alpha}{s_b}) \tag{12}$$

and

$$I_2 = (\sqrt{2\pi}s_b)\Phi(-\frac{A+\alpha}{s_b}). \tag{13}$$

Now, taking into account (12) and (13), from (10) we have

$$e^{rT}C_0 = S_0 exp(s_b^2/2 - \alpha)\Phi(\frac{s_b^2 - A - \alpha}{s_b}) - K\Phi(-\frac{A + \alpha}{s_b}). \tag{14}$$

Since from (11) one can see that $(\frac{s_b^2 - A - \alpha}{s_b}) = \frac{log(S_0/K) + (r + \sigma^2/2)T + (s_b^2 - \sigma^2 T)}{s_b}$ and $-\frac{A+\alpha}{s_b} = \frac{log(S_0/K) + (r - \sigma^2/2)T}{s_b}$, the statement of the theorem immediately follows from (14). Q.E.D.

References

Black, F. and Scholes, M. Pricing of options and Corporate Liabilities, J. Political Econ., 81, 637-654.

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