

## An Integrated Model of Market and Limit Orders\*

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We develop an integrated model in which a risk-neutral informed trader optimally chooses any combination of a market buy, a market sell, a limit buy including the limit buy price, and a limit sell including the limit sell price. Limit orders undercut the market maker and generate transactions inside the bid-ask spread. The informed trader exploits limit orders by submitting market orders even when the terminal value is inside the spread. When the terminal value is above the bid, a combined market buy-limit sell is more profitable than a market buy only. We obtain an analytic solution. *Journal of Economic Literature* Classification Numbers: D40, D82, G12, G14. © 1995 Academic Press, Inc.

### 1. INTRODUCTION

Market orders and limit orders are the dominant instruments for trading on most security exchanges.<sup>1</sup> For example, in the last six months of 1993 the New York Stock Exchange (NYSE) Superdot system processed 11.7

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<sup>1</sup> A market order is a request to buy (or sell) a fixed number of shares at the current price for buying (or selling). It yields a certain quantity, but an uncertain price. A limit order is a request to buy (or sell) a fixed number or up to a fixed number of shares at a limit buy (sell) price set by the limit order submitter. It yields a certain price, but an uncertain quantity.

million market orders and 19.3 million limit orders.<sup>2</sup> Yet until recently most models of market microstructure assumed that only market orders could be used for trading. In this paper, we develop an integrated model in which a risk-neutral informed trader can choose optimal quantities of four different kinds of orders to submit: a market buy (MB), a market sell (MS), a limit buy (LB) including the optimal limit buy price, and a limit sell (LS) including the optimal limit sell price.

Glosten and Milgrom (1985) (hereafter GM) develop an adverse selection model in which privately informed traders submit either a MB or a MS in order to optimally exploit their information. In GM, a market maker quotes a bid price and an ask price. Then, one agent per round can submit a MB or a MS for a unit quantity. A risk-neutral informed trader optimally chooses a *bang-bang* strategy. That is, one unit is bought (sold) when the private, expected value of the security is above the ask (below the bid) and zero units are traded when the expected value is inside the bid-ask spread.<sup>3</sup> The two parties in each transaction are an outside trader and the market maker. All transactions take place at either the bid or the ask.

Our model extends the GM framework to include limit orders. In contrast with GM, many of the trades in our model do not involve the market maker, but instead result from limit orders crossing with market orders. Traders set limit buy prices *below* the ask and limit sell prices *above* the bid (otherwise they would never execute in our single-period model). Thus, limit orders generate transactions inside the spread. These results correspond to what is observed in practice. For example, the 1993 NYSE Fact Book reports that 82.9% of NYSE transactions do not involve the market maker. Further, McNish and Wood (1992) document "hidden" limit orders crossing with market orders at prices inside the spread and Shapiro (1993) reports that when the spread is greater than  $\frac{1}{8}$ , 66% of NYSE trades occur inside the spread.

The presence of limit orders inside the spread creates an additional opportunity for the informed trader to exploit. We capture this feature of the informed trader's strategy. Under minimal distributional assumptions, we find that the informed trader sometimes chooses to submit market orders when the terminal value of the security is *inside* the bid-ask spread. To illustrate this, consider a MB submitted by the informed trader. Sometimes it executes at a price below the ask by crossing with a LS submitted by an uninformed trader<sup>4</sup> and it never executes above the ask. Thus the expected execution price is below the ask. Hence, there are some terminal

<sup>2</sup> These figures are for placed, not executed, orders. We thank Colin Moriarty of the NYSE for these figures.

<sup>3</sup> Admati and Pfleiderer (1989) extend the GM model to allow many risk-neutral, informed traders to execute market orders in the same round. The optimal strategy remains *bang-bang*.

<sup>4</sup> Our model also permits an informed MB to cross with an uninformed MS inside the spread.

values above the expected execution price and below the ask for which it is profitable for the informed trader to submit a MB.<sup>5</sup>

We also find a mutually beneficial interaction between opposing order types. For example, consider a combined MB and LS submission. Sometimes the two orders will cross and the informed trader will end up with a net zero trade. But this is actually a good thing. Even if the MB makes a profit on average, there are some states in which it will execute at a loss. Careful choice of the LS price can eliminate some of the loss states. Hence, we find that any time the terminal value of the security is above the bid, a combined MB-LS is more profitable than a MB alone. This intuition of limit orders acting as a "safety net" for market orders in the opposite direction is not one that has been captured in the microstructure literature.

We also obtain an analytic solution under special distributional assumptions. Substituting specific numerical values into the analytic solution, we provide further intuition of how the model works and explore properties of the equilibrium.

Kyle (1985) develops the second major branch of the adverse-selection literature. GM and Kyle differ primarily in their trading mechanism. GM is a *quote-driven* system where market makers post bid and ask prices before orders are submitted. Kyle's is an *order-driven* system where traders submit orders before prices are determined.<sup>6</sup> Most real-world trading systems are a hybrid of these simpler structures. For example, the New York Stock Exchange operates more like a quote-driven system for low-volume stocks and more like an order-driven, continuous auction system for high-volume stocks.

Our model builds on the GM quote-driven framework. For simplicity, we limit our analysis to a single-period setting in which (1) market makers set quotes first and then (2) market orders and limit orders arrive simultaneously. By contrast, much of the rapidly growing literature on limit orders has added limit orders to order-driven frameworks. Typically, in these models, (1) limit orders arrive, then (2) market orders arrive, and then (3) prices are determined. Comparing the two approaches, the benefit of our quote-driven approach over an order-driven approach is that it captures the interaction between limit orders and precommitted bid-ask quotes. However, the limitation of our approach is that it does not capture the effect of preexisting limit orders. How restrictive this limitation is depends in part on the cost of canceling old limit orders and resubmitting new ones. For example, if it costs nothing to cancel and resubmit limit orders, then

<sup>5</sup> Chordia and Subrahmanyam (1992) explain trading inside the bid-ask spread. They show that competitive market makers may follow a mixing strategy that randomly trades a certain percentage of market orders inside the spread, where the mixing percentage is a function of the order size.

<sup>6</sup> See Madhavan (1992) for a comparison of these two trading mechanisms.

a multiperiod limit order model could view all limit orders as being routinely canceled and resubmitted, with some new orders being resubmitted at the same price as old ones. Such a multiperiod model would be isomorphic to a repeated single-period model. However, if the cost (including the opportunity cost) of canceling and resubmitting is substantial, then our single-period setting would be restrictive. Other limitations of our model include: (1) all uninformed orders are completely exogenous, (2) there are no innovations in public information between the time that limit orders are submitted and the time they execute (see Brown and Holden (1994) for an analysis of this issue), (3) the market makers do not get to use limit orders on the book to extract any information, and (4) the "free option" problem raised by Rock (forthcoming) (see footnote 7) is not addressed.

The limit order literature has grown rapidly in the last few years. The earliest model was Kyle (1989), which defines a "schedule of limit orders" as a linear demand schedule covering all quantities. Rock (forthcoming) models limit orders as a request for a specific quantity of the security at a specific price. However, his model restricts submission of limit orders to uninformed traders only, and both types of orders are limited to unit quantities.<sup>7</sup> Angel (1990), Easley and O'Hara (1991),<sup>8</sup> Foucault (1993), and Harris (1994) model an informed investor's order placement strategy for choosing between market and limit orders. But all of these models require investors to choose one or the other—never both market and limit orders. Kumar and Seppi (1993) develop a model of both market and limit orders based on an order-driven trading mechanism. Among other results, they show that brokerage costs have important influence on the depth and composition of the limit order book. They prove the existence of an equilibrium using a fixed-point theorem. However, they do not provide an analytic solution. Glosten (1994) develops a model of an "electronic exchange" in which a large number of risk-neutral liquidity suppliers submit limit orders. A single risk-averse, utility-maximizing trader, who may or may not be informed, arrives at any point in time. The trader submits a market order only that is executed against the limit order book. The trader is not permitted to submit a limit order. Glosten analyzes the properties of the electronic exchange and demonstrates that it provides as much liquidity as possible in extreme situations. He also shows that the electronic exchange cannot be undercut by an entering exchange that earns positive expected profits. Black (1991, 1994) develops a series of conjectures about the likely character of future trading mechanisms in equilibrium. He conjectures that unin-

<sup>7</sup> Rock focuses on how the market maker's inventory problem is affected by uninformed limit orders. His key result is that uninformed limit orders exacerbate the inventory problem and thus delay the full adjustment to an inventory shock. By contrast, our model focuses on the adverse selection problem.

<sup>8</sup> This paper focuses on stop orders.

formed traders will use "unpriced limit orders" that specify the number of shares requested and a level of urgency. Greater urgency implies a higher average percentage execution rate and greater losses to informed traders and vice versa. He conjectures that informed traders will predominantly use market orders, but may occasionally use unpriced limit orders. He argues that uninformed traders cannot do better using (1) conventional limit orders with a limit price, (2) specialized exchanges, (3) basket trading, or (4) sunshine trading.

The plan of the paper is as follows. Section 2 develops the model under minimal distributional assumptions, characterizes the informed trader's optimal strategy, and then develops an analytic solution under specific distributional assumptions. Section 3 provides a numerical illustration of the model. Section 4 concludes. All proofs are in Appendixes A and B.

## 2. THE MODEL

### 2.1. General

There are three classes of economic agents: one informed trader, two or more uninformed traders, and  $N \geq 2$  identical, competitive market makers. All agents are assumed to be risk neutral. The model is a single-period, two-date model with one risky asset. On date 0, first the market makers quote prices, then the informed and uninformed traders submit their market and limit orders simultaneously, and finally all orders are executed (or not executed) according to the protocol of the security exchange. On date 1, the terminal value is realized and all agents receive this value of the security. For most of this section, the model is analyzed without distributional assumptions other than the support of the random variables. In Section 2.5, additional distributional assumptions are invoked in order to obtain an analytic solution, and in Section 3, numerical values are substituted into the analytic solution in order to illustrate the model.

The security exchange is assumed to follow a quote-driven protocol. Specifically, on date 0 each market maker declares a *bid-ask quote*. The individual market maker is committed to trade on the opposite side of each MB at the quoted ask and to trade on the opposite side of each MS at the quoted bid. In equilibrium, there is a single competitive ask price  $a$  and a single competitive bid price  $b$ , each of which yields zero expected profits for each market maker.

The informed trader observes the liquidation value of a security  $v$ , where  $v$  is a real value from the bounded interval  $[v_1, v_H]$  and  $\mu$  denotes the unconditional mean  $E[v]$ . Then, the informed trader makes a choice of orders to submit. The informed trader's choice set is summarized in Panel

TABLE I  
OVERVIEW OF CHOICE VARIABLES AND RANDOM VARIABLES IN THE MODEL.

A: The Informed Trader's Choice Set		
Order	Choice variable	Space of the choice variable
Market buy (MB)	$MB^I$ = number of MB shares	$\{0, 1, 2, \dots, Q\}$
Market sell (MS)	$MS^I$ = number of MS shares	$\{-Q, \dots, -2, -1, 0\}$
Limit buy (LB)	$N_{LB}^I$ = number of LB shares $b^I$ = LB price	$\{0, 1, 2, \dots, Q\}$ $[v_L, v_H]$
Limit sell (LS)	$N_{LS}^I$ = number of LS shares $s^I$ = LS price	$\{-Q, \dots, -2, -1, 0\}$ $[v_L, v_H]$

B: Uninformed Traders' Submissions		
Order	Random variable	Support of the distribution
Market buy (MB)	$MB^U$ = number of shares	$\{0, 1, 2, \dots, Q\}$
Market sell (MS)	$MS^U$ = number of shares	$\{-Q, \dots, -2, -1, 0\}$
Limit buy (LB)	$N_{LB}^U$ = number of shares $b^U$ = limit buy price	$\{0, 1, 2, \dots, Q\}$ $[b, \mu]$
Limit sell (LS)	$N_{LS}^U$ = number of shares $s^U$ = limit sell price	$\{-Q, \dots, -2, -1, 0\}$ $[\mu, a]$

Note.  $Q$  is the maximum number of shares that can be bought or sold,  $v_L$  is the lower bound of  $v$ ,  $b$  is the bid price,  $\mu$  is the unconditional mean of  $v$ ,  $a$  is the ask price, and  $v_H$  is the upper bound of  $v$ .

A of Table I. The first column lists the four types of orders that the informed trader is permitted to submit, a MB, a MS, a LB, and a LS. For each order, the informed trader chooses the corresponding number of shares to submit,  $MB^I$ ,  $MS^I$ ,  $N_{LB}^I$ , and  $N_{LS}^I$ . Throughout this paper we follow the following conventions:

- buy orders are nonnegative integers from the set  $\{0, 1, 2, \dots, Q\}$ , where  $Q$  is the upper bound on number of shares that may be bought,
- sell orders are nonpositive integers from the set  $\{-Q, \dots, -2, -1, 0\}$ , and
- an I superscript refers to the informed trader and U superscript refers to uninformed traders.

For the limit orders, the informed trader chooses the corresponding limit prices  $b^I$  and  $s^I$  from the interval  $[v_L, v_H]$ . Limit orders live for one trading date; unless executed on that date, they expire.

Simultaneously, the uninformed traders trade for liquidity or other reasons that are not modeled. The uninformed traders' submissions are summarized in Panel B of Table I. Collectively, they submit the same types of orders as the informed trader. For each order, a random variable determines the corresponding number of shares they submit,  $MB^U$ ,  $MS^U$ ,  $N_{LB}^U$ , and  $N_{LS}^U$ . For the limit orders, one random variable determines the

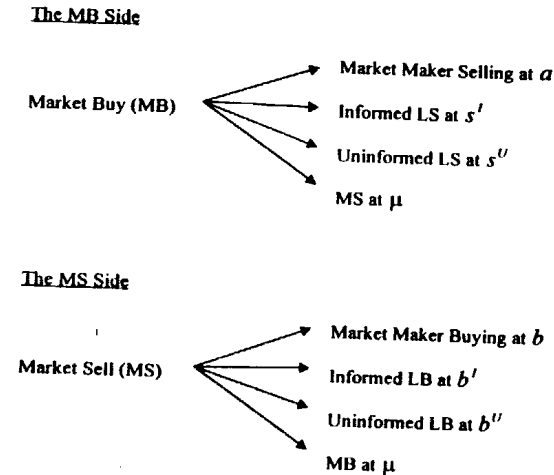


FIG. 1. Order execution on the two sides of the market. The MB side. The four ways a MB can execute and the corresponding transaction price for each way. The MS side. The four ways a MS can execute and the corresponding transaction price for each.  $a$  is the ask price,  $s^I$  is the informed limit sell price,  $s^U$  is the uninformed limit sell price,  $\mu$  is the unconditional mean of the risky asset,  $b$  is the bid price,  $b^I$  is the informed limit buy price, and  $b^U$  is the uninformed limit buy price.

limit buy price  $b^U$  from the interval  $[b, \mu]$  and another random variable determines the limit sell price  $s^U$  from the interval  $[\mu, a]$ .<sup>9</sup> All of the underlying random variables,  $v$ ,  $MB^U$ ,  $MS^U$ ,  $N_{LB}^U$ ,  $N_{LS}^U$ ,  $b^U$ , and  $s^U$  are assumed independent of each other.

2.2. Order Execution

Many possible combinations of orders may arrive simultaneously at the security exchange. There are two sides of the security exchange for executing orders. We call one side the *market buy side (MB side)* and the other the *market sell side (MS side)*. Figure 1 illustrates these two sides.

Figure 1 shows that on the *MB side*, a MB order can execute in four ways: (1) against a market maker who is selling, (2) against an informed LS, (3) against an uninformed LS, and (4) against a MS. Similarly, Fig. 1 shows that on the *MS side* there are four analogous ways for a MS order to execute. In other words, a MB can trade with four different instruments of selling and a MS can trade with four different instruments of buying. Hence, the two sides are disconnected from each other, except in the special

<sup>9</sup>Note that a limit sell price above the ask  $a$  does not make sense in our setting since it would not be executed. Similarly, a limit buy price below the bid  $b$  would not be executed.

case when a MB and a MS trade with each other. For this special case, we adopt the convention that MB and MS cross at the unconditional mean  $\mu$ . Since the unconditional mean is an exogenous value, the system of simultaneous equations that determine endogeneous prices and trading strategies on the MB side and the system of simultaneous equations on the MS side are effectively disconnected from each other. We exploit this disconnection by sorting the informed trader's objective function into two parallel profit functions, one on the MB side and the other on the MS side. Similarly, we sort the market maker's objective function into one on the MB side and the other on the MS side.<sup>10</sup>

We wish to emphasize that we are not restricting the informed trader to trading on only one side. Indeed, we show for some realizations that it is optimal for the informed trader to submit orders on both the MB side and the MS side simultaneously. In other words, the total equilibrium is the sum of the two parts.

The protocol of a security exchange specifies the rules for executing any combination of orders that arrive at the exchange. In our model, the protocol is based on two rules. The first is for executing a market order and is called "price priority." First, the market buy (MB) orders are matched with the lowest price that can be obtained from any of the four selling instruments. If another MS exists, the lowest price is  $\mu$ . If no MS exists, but a limit sell (LS) exists, then the lowest price is either  $s^1$  or  $s^{11}$ . If neither an MS nor a LS exists, then the lowest price is the competitive ask  $a$ . Next, any remaining shares from the MB orders move up the schedule of the limit order book and are matched with the next lowest price that can be obtained from the selling instruments, and so on.<sup>11</sup> The protocol works in an analogous manner on the MS side.

The second rule of the protocol specifies that market orders will obtain pro rata rationing of limit order prices. To illustrate this, Table II gives an example in which the informed trader has submitted a MB for 8 shares and a LS for 7 shares at \$49. Uninformed traders have submitted a MB for 4 shares and a LS for 9 shares at \$48.

The total shares requested by the informed MB and the uninformed MB is 12 shares. Initially both MBs execute against the uninformed LS, at \$48 rather than \$49. But since the uninformed LS is only for 9 shares, they are pro rata rationed to the LBs as follows:

- $(\frac{8}{12}) \times 9$  shares = 6 shares go to the informed MB at \$48 and
- $(\frac{4}{12}) \times 9$  shares = 3 shares go to the uninformed MB at \$48.

<sup>10</sup> The MB side should not be confused with MB orders. The MB side includes both MB orders and LS orders (which might execute with each other). Similarly, the MS side includes both MS orders and LB orders.

<sup>11</sup> This clearing procedure closely matches the actual procedure of the NYSE. See NYSE Constitution and Rules (1992), Rule 72.

TABLE II  
EXAMPLE OF PRO RATA RATIONING

	Better price: Uninformed LS for 9 shares at \$48	Worse price: Informed LS for 7 shares at \$49	Total shares and weighted average price
Informed MB for 8 shares	6 shares at \$48	2 shares at \$49	8 shares at \$48.25
Uninformed MB for 4 shares	3 shares at \$48	1 share at \$49	4 shares at \$48.25
Total	9 shares at \$48	3 shares at \$49 and 4 LS shares unexecuted	12 shares at \$48.25 and 4 LS shares unexecuted

Note. In this example, the informed trader has submitted a MB for 8 shares and a LS for 7 shares at \$49. Uninformed traders have submitted a MB for 4 shares and a LS for 9 shares at \$48.

This satisfies nine of the shares requested by the MBs. The remaining three shares are obtained by moving up the limit order book to the informed LS at \$49. Since the informed LS is for 7 shares, the remaining 3 shares requested by the MBs are crossed with the informed LS at \$49. In summary, the informed MB for 8 shares received a weighted average price of \$48.25 and the uninformed MB for 4 shares received the same weighted average price of \$48.25.

### 2.3. The Informed Trader's Problem

Recall the basic distinction between order types: (1) a market order trades a certain number of shares at an uncertain price, and (2) a limit order trades an uncertain number of shares at a certain price. For an MB submitted by the informed trader, let  $MB^1$  be the certain number of shares and  $\tilde{p}_{MB}$  be the uncertain price. Similarly, for a market sell by the informed, let  $MS^1$  be the certain number of shares and  $\tilde{p}_{MS}$  be the uncertain price. For a limit buy by the informed, let  $L\tilde{B}^1$  be the uncertain number of shares that actually get executed and let  $b^1$  be the certain price. Similarly, for a limit sell by the informed, let  $L\tilde{S}^1$  be the uncertain number of shares that actually get executed and let  $s^1$  be the certain price. The probability distribution for the random number of shares executed under an informed limit buy is a function of the number of shares requested ( $N_{LB}^1$ ) and the limit buy price,  $L\tilde{B}^1 = L\tilde{B}^1(N_{LB}^1, b^1)$ . Similarly, the probability distribution for the random number of shares executed under an informed limit sell is a function of the number of shares requested ( $N_{LS}^1$ ) and the limit sell price,  $L\tilde{S}^1 = L\tilde{S}^1(N_{LS}^1, s^1)$ .

The informed trader's objective function sorts into two pieces. Let  $J_{MB}$  be the conditional expected profit function given the terminal value  $v$  for the MB side and let  $J_{MS}$  be the same function for the MS side. The informed trader maximizes  $J_{MB}$  by choosing the optimal MB and LS and maximizes  $J_{MS}$  by choosing the optimal MS and LB,

$$\begin{aligned} \text{Max } J_{MB} + J_{MS} \\ = \text{Max } E[MB^1(v - \bar{p}_{MB}) + L\bar{S}^1(v - s^1) | v] \\ + E[MS^1(v - \bar{p}_{MS}) + L\bar{B}^1(v - b^1) | v]. \end{aligned} \quad (1)$$

Since the informed trader knows the terminal value  $v$ , the expectations operator can be passed through to obtain

$$\begin{aligned} \text{Max } MB^1(v - E[\bar{p}_{MB}]) + E[L\bar{S}^1](v - s^1) \\ + MS^1(v - E[\bar{p}_{MS}]) + E[L\bar{B}^1](v - b^1). \end{aligned} \quad (2)$$

This form of the informed trader's objective function highlights the fact that the market order decisions rest on their expected prices  $E[\bar{p}_{MB}]$  and  $E[\bar{p}_{MS}]$  and that the limit order decisions rest on their expected quantities  $E[L\bar{B}^1]$  and  $E[L\bar{S}^1]$ .<sup>12</sup>

We now proceed to characterize the informed trader's optimal choices, without distributional assumptions other than the support of the random variables. The optimal limit sell price is determined by the First Order Condition (F.O.C.),  $\partial J_{MB}/\partial s^1 = 0$ , and the optimal limit buy price is determined by the F.O.C.,  $\partial J_{MS}/\partial b^1 = 0$ . The following lemma characterizes the optimal limit sell price  $s^{1*}$  and optimal limit buy price  $b^{1*}$ , where the "\*" superscript designates an optimal value.

LEMMA 1. *Optimal limit prices meet the following conditions:*

1.  $s^{1*} < a$ .
2.  $s^{1*} > v$ .
3.  $s^{1*} \geq \mu$ .
4.  $b^{1*} > b$ .
5.  $b^{1*} < v$ .
6.  $b^{1*} \leq \mu$ .

Essentially, Lemma 1 rules out the ranges of limit buy prices and limit sell prices that would yield nonpositive expected profits.

<sup>12</sup> For convenience, we suppress the notation for the conditioning on  $v$ .

The expected price of a market buy,  $E[\bar{p}_{MB}]$ ,<sup>13</sup> is based on the multiple ways for a market buy to execute. Figure 1 illustrates four ways. One of these is against an uninformed limit sell. This way can be split into two special cases: (1) when  $s^U < s^1$  or (2) when  $s^U \geq s^1$ . In total, there are five ways for a market buy to execute and each way has its corresponding price. A market buy can trade: (1) with the dealer at the competitive ask  $a$ , (2) with the informed trader's limit sell at  $s^1$ , (3) with the uninformed trader's limit sell at a random price  $s^U$  that is lower than the informed trader's price ( $s^U < s^1$ ), (4) with the uninformed trader's limit sell at a random price  $s^U$  that is weakly higher than the informed trader's price ( $s^U \geq s^1$ ),<sup>14</sup> and (5) with a market sell at  $\mu$ . Let  $E[s^U | s^U < s^1]$  denote the expected value of  $s^U < s^1$  and let  $E[s^U | s^U \geq s^1]$  denote the expected value of  $s^U \geq s^1$ . Then, the expected price of a market buy must be a weighted combination of the corresponding five expected prices

$$E[\bar{p}_{MB}] = \pi_1 a + \pi_2 s^1 + \pi_3 E[s^U | s^U < s^1] + \pi_4 E[s^U | s^U \geq s^1] + \pi_5 \mu,$$

where  $\pi_1, \dots, \pi_5$  are the probabilities of each of the five events. Similarly, the expected price of a market sell can be written as a weighted combination of five analogous events. The following lemma identifies a particular interval that the expected price of a MB (or MS) must lie within. It is based upon the fact that the expected price is a weighted average of five prices, each of which lies in the same interval, and there is a nonzero weight on both endpoints.

LEMMA 2.

1. *The expected price of a market buy  $E[\bar{p}_{MB}] \in (\mu, a)$  and*
2. *the expected price of a market sell  $E[\bar{p}_{MS}] \in (b, \mu)$ .*

Whenever a MB trades below the ask price, the difference is commonly called *price improvement*.<sup>15</sup> Any source of expected price improvement will lead to an expected price that is inside the spread.<sup>16</sup> Our model captures

<sup>13</sup> Clearly,  $E[\bar{p}_{MB}]$  is not defined for  $MB^1 = 0$ , and  $E[\bar{p}_{MS}]$  is not defined for  $MS^1 = 0$ . The discussion following is based on nonzero market orders.

<sup>14</sup> This event happens when the number of shares demanded by MB orders is greater than the number of shares requested in the informed trader's LS order, and thus some residual number of shares are traded against the uninformed LS order even though it is a worse price.

<sup>15</sup> Analogously, whenever an MS trades above the bid price, the difference is price improvement.

<sup>16</sup> Lee (1992) finds empirically that when the bid ask quote is \$1/4 or more, the NYSE delivers price improvement on more than 61% of all trades and Cincinnati delivers price improvement on more than 78% of all trades. Peterson and Fialkowski (1992) find in their sample that 50% of trades on all exchanges receive price improvement.

two sources of expected price improvement. First, a MB can cross with a MS at  $\mu$ , which is inside the spread. Secondly, a MB can cross with a LS at a limit sell price which is inside the spread.<sup>17</sup> The expected price improvement available in our framework brings into play the spread, as shown in the proposition below.

PROPOSITION 1. *Market orders submitted inside the spread.*

1. *There exist values of  $v < a$  such that it is optimal to submit a market buy ( $MB^I > 0$ ).*
2. *There exist values of  $v > b$  such that it is optimal to submit a market sell ( $MS^I < 0$ ).*

Intuitively, it is optimal to submit a positive MB any time that  $v$  is greater than the expected price of a MB ( $E[\bar{p}_{MB}]$ ). From Lemma 2, we know that the expected price of a MB is less than the ask price  $a$ , due to the expected price improvement. Thus when  $a > v > E[\bar{p}_{MB}]$ , it is optimal to submit a positive MB. This result stands in contrast to the GM framework with market orders only. In GM, there is no opportunity for price improvement and thus informed trading never takes place inside the spread.

The following lemma characterizes conditions under which each of the four order types can be ruled out.

LEMMA 3. *The following quantity choices are optimal:*

1. *When  $v > a$ ,  $N_{LS}^I = 0$ .*
2. *When  $v < b$ ,  $N_{LB}^I = 0$ .*
3. *When  $v > \mu$ ,  $MS^I = 0$ .*
4. *When  $v < \mu$ ,  $MB^I = 0$ .*

Lemma 3 provides the first sketch of the informed trader's overall strategy and is illustrated in Fig. 2. In the figure, the range of possible terminal values  $[v_L, v_H]$  is subdivided into four subintervals: (1)  $v \leq b$ , (2)  $\mu \geq v > b$ , (3)  $a > v > \mu$ , and (4)  $v \geq a$ . The first subinterval is at or below the bid and here the informed trader sells shares via LS and MS, but does not buy. Similarly, the fourth subinterval is at or above the ask and here the informed trader buys shares via LB and MB, but does not sell. The most interesting cases are the second and third subintervals when  $v$  is inside the

<sup>17</sup> McNish and Wood (1992) document "hidden limit orders" on the NYSE, which they define as a limit order with a limit price that is inside the spread, and the quote is not updated to reflect the better price offered by the limit order. When a hidden limit order executes, it is a source of price improvement relative to the nonupdated quotes. Using an aggressive 10 second rule, McNish and Wood classify about 50% of the limit orders in the TORQ database (obtained from the New York Stock Exchange) as hidden.

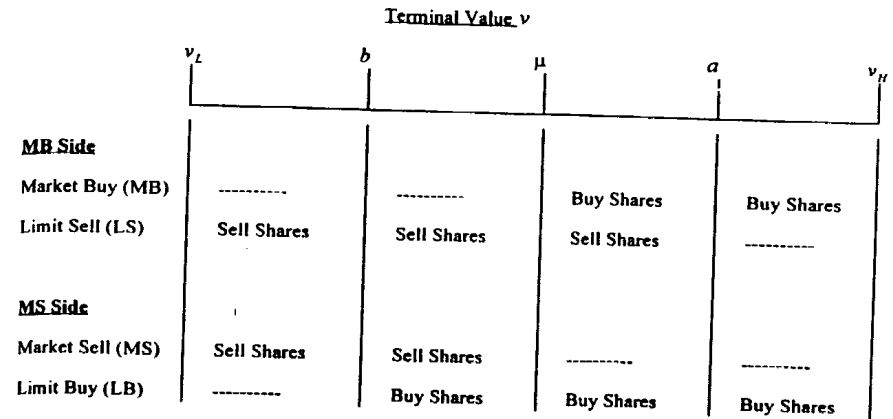


FIG. 2. Sketch of the informed trader's strategy by four subintervals of the terminal value  $v$ . Above the ask, the informed trader strictly buys; below the bid, the informed trader strictly sells. Inside the spread, the informed traders use a mix of instruments in order to submit buy and sell orders simultaneously.  $v_L$  is the lower bound of  $v$ ,  $b$  is the bid price,  $\mu$  is the unconditional mean of the risky asset,  $a$  is the ask price, and  $v_H$  is the upper bound of  $v$ .

bid and ask quotes. In these two subintervals, the informed trader may wish to submit both a limit buy and a limit sell. We will show shortly that this is the optimal policy for any value of  $v$  in these two subintervals. In addition, the informed trader may wish to submit a market buy order when  $v > \mu$  and a market sell order when  $v < \mu$ . We will show shortly that in some cases it is optimal to submit three orders simultaneously. Specifically, for some values in the second subinterval, it is optimal to submit:

- a LS on the MB side and
- a MS-LB on the MS side.

For some values in the third subinterval, it is optimal to submit:

- a MB-LS on the MB side and
- a LB on the MS side.

When the informed trader submits multiple orders simultaneously, it is possible for some of the orders to interact. For example, consider what might happen if a MB and a LS are submitted simultaneously on the MB side. They might execute with each other. By contrast, it is impossible for the LB on the MS side to execute with either order on the MB side. To distinguish these two cases, we will call two orders that may execute with

each other a "combined" order and an order that may not execute with others submitted simultaneously an "isolated" order. In this example, we have a combined MB-LS and an isolated LB.

In order to calculate the informed trader's expected profits, we must analyze both the combined case and the isolated case. The expected profits of an isolated LB is simply  $E[L\bar{B}^1](v - b^1)$ . Given any value for  $b^1$ , as long as  $(v - b^1)$  is positive, it is obviously advantageous to submit a large  $N_{1,B}^1$  in order to maximize the expected number of shares that will be executed  $E[L\bar{B}^1]$ . Since  $N_{1,B}^1$  has no effect on the probability of execution (just the maximum number of shares that can be bought), it is clearly optimal to submit the largest request possible  $N_{1,B}^1 = Q$ . Similarly, the optimal policy for an isolated LS is  $N_{1,S}^1 = -Q$ . This is the optimal policy for the combined case as well, as specified by the following lemma.

LEMMA 4. For combined submissions, we have:

1. For any value  $v < a$ , any  $MB^1$ , and any  $s^1 \in [\mu, a]$ , the optimal LS size is  $N_{1,S}^1 = -Q$ .
2. For any value  $v > b$ , any  $MS^1$ , and any  $b^1 \in [b, \mu]$ , the optimal LB size is  $N_{1,B}^1 = Q$ .

Lemma 4 says that when an informed trader can earn positive expected profits on a limit order of any size, then it is optimal to request the maximum number of shares (in absolute value). Intuitively, this is because there is no price impact of increasing the number of shares requested.

Now consider a specific example of a combined MB-LS. Suppose that the informed trader observes a terminal value  $v = \$52$  and computes an optimal limit sell price  $s^1 = \$53$ . Any time the LS executes, it earns \$1. Suppose that when the uninformed LS arrives, half the time the limit sell price  $s^U = \$51$  and half the time  $s^U = \$54$ . When an uninformed LS arrives at \$51, the MB crosses with it and makes \$1. When an uninformed LS arrives at \$54, the MB crosses with the informed LS at \$53, since this is a better price than \$54. The MB loses \$1, but this loss is smaller than the loss that would have happened without the informed LS. Without the informed LS, the MB would have crossed with an uninformed LS at \$54 and lost \$2.<sup>18</sup> Hence, the combined MB-LS is strictly more profitable than an isolated MB, because the LS reduces MB losses in a few states. The more shares that the informed LS requests, the bigger the total loss on the MB that can be avoided.

This leads to one of the most interesting consequences of including both market and limit orders in the same framework.

<sup>18</sup> If an uninformed limit sell price of \$51 is more likely than \$54, then the MB component of the combined MB-LS would yield a positive profit on average even though it loses money in some bad states. Hence it is still optimal to submit the MB.

PROPOSITION 2: Combined Market Order and Limit Order Submissions.

1. When  $v < a$ , the informed trader's expected profits are strictly greater for a combined MB-LS submission than for a MB submission alone.
2. When  $v > b$ , the informed trader's expected profits are strictly greater for a combined MS-LB submission than for a MS submission alone.

This proposition says the option-like, conditional execution of limit orders enhances the profitability of market orders by reducing their losses in bad states. More generally, it suggests that it is optimal to fully exploit the richer menu of combined strategies that are available when security exchanges permit both market and limit orders.

In analyzing the model we focus on monotonic equilibria, which are defined as equilibria for which the informed trader chooses a monotonic strategy.<sup>19</sup>

DEFINITION: Monotonic strategy. A strategy is *monotonic* when:

- on the MB side,
  - for any ask price  $a$  and for any limit sell price  $s^1$ , the informed trader chooses a weakly larger number of MB shares,  $MB^1$ , for larger  $v$ , and
  - for any ask price  $a$  and for any number of MB shares,  $MB^1$ , the informed trader chooses a weakly smaller limit sell price for larger  $v$ ; and
- on the MS side,
  - analogous conditions to those above.

A monotonic strategy divides up the set of possible terminal values  $v$  into series of subintervals. Each subinterval is based on a pair of optimal values for the MB quantity and the LS price ( $MB^1, s^1$ ). The bound between two adjacent subintervals can be described by a critical value, where the informed trader is just indifferent between the profit generated by alternative MB-LS combinations. Let the critical value  $v_k$  be the value of  $v$  where the informed trader is indifferent between:

- a combination of  $MB^1 = k$  shares and the corresponding optimal limit sell price  $s_k^1$ , versus
- an alternative combination of  $MB^1 = j$  shares ( $j > k$ ) and the corresponding optimal limit sell price  $s_j^1$ .

The alternative combination is the profit maximizing combination selected from the set of feasible values of  $MB^1 > k$ . The critical value  $v_k$  is defined using the profit functions from Eq. (1) as

<sup>19</sup> Intuitively, it seems unlikely that a nonmonotonic strategy would be optimal, but we can't rule this possibility out.

$$J_{MB}(v_k, MB^1 = k, s_k^1) = \text{Max} \{J_{MB}(v_j, MB^1 = j, s_j^1) \text{ for all } j > k\}^{20} \quad (3)$$

Critical values on the MS side are defined in an analogous manner.

#### 2.4. The Market Maker's Problem

The market maker is responsible for being the trader of last resort for the market orders. For example, after a MB has crossed with MS and LS orders, any residual shares would trade with the market maker. Let  $X$  be the total number of shares that all market makers sell at the competitive ask price on their own account. The market clearing condition for MB orders is that  $X = -(\text{residual number of shares requested by MB orders})$ . A similar market clearing condition holds on the MS side.

The equilibrium ask price  $a$  is determined by an equilibrium condition that each of the risk-neutral, market makers earns zero expected profit on the MB side. This condition is given by

$$E \left[ \left( \frac{X}{N} \right) (v - a) \right] = 0. \quad (4)$$

The bid price is determined by a similar zero expected profit condition on the MS side. This completes the overview of the model and leads to the following definition of equilibrium.

**DEFINITION: Equilibrium.** Equilibrium prices and quantities are:

1. on the MB side,
  - an ask price  $a$ ,
  - a monotonic set of critical values  $v_k$  for  $k = 0, \dots, Q - 1$ , and
  - a limit sell price function  $s^1 = s^1(k, v)$

that satisfy the following system of simultaneous equations:

- the equilibrium condition for the ask  $E[(X/N)(v - a)] = 0$ ,
  - the informed trader's indifference equation (Eq. 3) for all  $k$ , and
  - the F.O.C.  $\partial J_{MB} / \partial s^1 = 0$ ;
2. on the MS side,
    - a bid price  $b$ ,
    - a monotonic set of critical values  $v_k$  for  $k = -Q, \dots, -1$ , and
    - a limit buy price function  $b^1 = b^1(k, v)$

<sup>20</sup> To complete the set of critical values on the MB side, we add the highest and lowest bounds,  $v_{-1} = v_L$  and  $v_Q = v_H$ .

that satisfy the analogous system of simultaneous equations for the MS side.

#### 2.5. An Analytic Solution

Under the above systems of simultaneous equations, it is impossible to determine if a solution exists or is unique. However, in one special case we can obtain an analytic solution as shown in the following proposition.

**PROPOSITION 3: Analytic Solution.** Under special distributional assumptions (see Appendix A), the unique monotonic equilibrium is given below.

1. On the MB side, the ask price is

$$a = \frac{-d - \sqrt{d^2 - 4ce}}{2c},$$

where the coefficients are given by

$$c = \sum_{k=0}^Q (\alpha_k - \alpha_{k-1})(\alpha_k + \alpha_{k-1} - 2)E[\tilde{X} | MB^1 = k],$$

$$d = \sum_{k=0}^Q (\alpha_k - \alpha_{k-1})(\beta_k + \beta_{k-1}) + (\beta_k - \beta_{k-1})(\alpha_k + \alpha_{k-1} - 2)E[\tilde{X} | MB^1 = k],$$

$$e = \sum_{k=0}^Q [(\beta_k - \beta_{k-1})(\beta_k + \beta_{k-1})]E[\tilde{X} | MB^1 = k],$$

and the monotone set of critical values are  $v_k = \alpha_k a + \beta_k$  for  $k \in \{0, \dots, Q - 1\}$ , where expressions for  $\alpha_k$  and  $\beta_k$  are given in Appendix B.

2. On the MS side, analogous equations specify the equilibrium values.

In the following section we substitute specific numerical values into this analytic solution in order to provide further intuition about how the model works and to explore the properties of the equilibrium.

### 3. A NUMERICAL ILLUSTRATION OF THE MODEL

This section provides a numerical example to illustrate the operation of the model. We assume that

- $v$  has a continuous uniform distribution over  $[\$40.00, \$50.00]$ , (thus  $\mu = E[v] = \$45.00$ ),

- $Q = 10$ ,
- $N_{1,S}^U = \begin{cases} -10 & \text{with probability} = .9, \\ 0 & \text{with probability} = .1, \end{cases}$
- $N_{1,B}^U = \begin{cases} 10 & \text{with probability} = .9, \\ 0 & \text{with probability} = .1, \end{cases}$
- $s^U = \$45.50$ , which is a constant in the interval ( $\mu = \$45.00$ ,  $a = \$47.63$ ), and
- $b^U = \$44.50$ , which is a constant in the interval ( $b = \$42.37$ ,  $\mu = \$45.00$ ).

The informed trader's strategy on the MB side is a combined MB-LS. Holding the LS part fixed, the optimal MB quantity is determined by calculating a series of critical values  $v_k$ , where the informed trader is exactly indifferent between submitting a MB for one particular quantity versus another. Using Eq. (3), we obtain the following critical values:

- $v_0 = \$45.12$ , which is the indifference point between  $MB^I = 1$  share and  $MB^I = 0$  shares;
- $v_1 = \$45.15$ , which is the indifference point between  $MB^I = 2$  shares and  $MB^I = 1$  share;
- $v_2 = \$45.20$ , which is the indifference point between  $MB^I = 3$  shares and  $MB^I = 2$  shares; etc.

Thus for any value of  $v \in (\$45.12, \$45.15]$ , the optimal MB is one share, for any value of  $v \in (\$45.15, \$45.20]$ , the optimal MB is two shares, etc.

Turning to the LS part of the combined MB-LS, we focus on the optimal limit sell price  $s^I$ . The assumption that the uninformed trader's limit sell price is constant ( $s^U = \$45.50$ ) simplifies matters a great deal. If the informed trader sets  $s^I$  in the range of values  $\$45.00 \leq s^I < \$45.50$ , then it will undercut both the price of the market maker and the price of the uninformed LS. If the informed trader sets  $s^I$  in a second range of values  $\$45.50 \leq s^I < \$47.63$ , then it will undercut the price of the market maker, but will not undercut the price of the uninformed LS.<sup>21</sup> Within each of the two ranges, the profit function  $J_{MB}$  is strictly increasing in  $s^I$ . Intuitively, increasing  $s^I$  increases the profit per share when the LS executes without changing the probability that the LS will execute. Hence, the optimal limit sell price

<sup>21</sup> In order to avoid a cumbersome exposition that talks about undercutting the market maker's price "by an arbitrarily small amount," we adopt the tie-breaking rule that when the market maker and informed LS have both set the same price, then the informed LS gets the entire order flow. For the same reason, we adopt the tie-breaking rule that when the informed LS and the uninformed LS have both set the same price, then the informed LS gets the entire order flow.

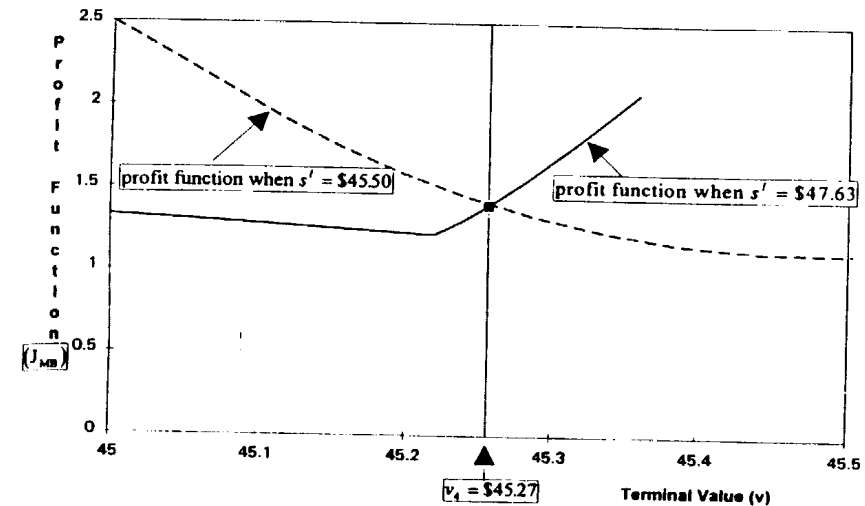


FIG. 3. Profit function ( $J_{MB}$ ) for two-limit sell prices. The profit function ( $J_{MB}$ ) for a combined MB-LS when  $s^I = \$45.50$  is downward sloping. The profit function for a combined MB-LS when  $s^I = \$47.63$  is upward sloping. They cross at the "switching value"  $v_4 = \$45.27$ , at which point the informed trader switches from  $s^I = \$45.50$  to  $s^I = \$47.63$ .

within each range is the maximum price in the range (i.e.,  $s^I = s^U = \$45.50$  in the first range and  $s^I = a = \$47.63$  in the second range).

Intuitively, the tradeoff between these two values is that  $s^I = \$45.50$  will lead to less profit per share when the informed LS executes, but it will execute more often since the price is better than the uninformed LS. Alternatively, setting  $s^I = \$47.63$  will lead to more profit per share when the informed LS executes, but it will execute less often since the price is worse than the uninformed LS. The optimal choice depends on which one will yield the higher value of the profit function  $J_{MB}$  for a given realization of  $v$ . This choice is illustrated in Fig. 3 which graphs the informed trader's profit function  $J_{MB}$  for combined MB-LS as a function of the terminal value  $v$ . The downward sloping curve is  $J_{MB}$  for  $s^I = \$45.50$  and the upward sloping curve is  $J_{MB}$  for  $s^I = \$47.63$ . The two curves cross at  $v = \$45.27$ , at which price the informed trader's strategy switches from  $s^I = \$45.50$  to  $s^I = \$47.63$ . Formally, this switching value is the critical value,  $v_4$ , where the informed trader is indifferent between a combined MB for four shares and a LS at  $s^I = \$45.50$  versus the profit maximizing alternative combination, which turns out to be a MB for seven shares and a LS at  $s^I = \$47.63$ .

The informed trader's overall strategy is illustrated in Fig. 4 (which is an augmented version of Fig. 2). This strategy differs in three regions.

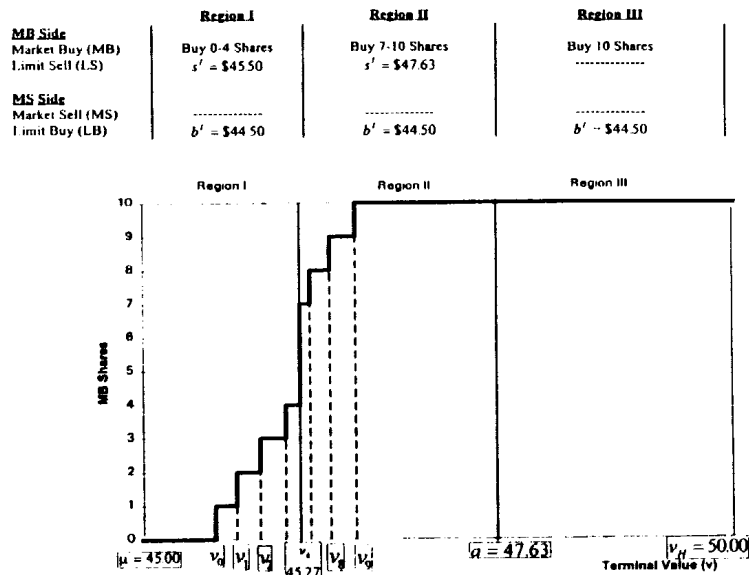


FIG. 4. The informed trader's overall strategy by three regions of the terminal value  $v$ . Table at the top shows the optimal orders to submit on the MB side and the MS side for three regions. Region I is  $v < \$45.27$ , Region II is  $\$45.27 < v < \$47.63$ , and Region III is  $v > \$47.63$ .  $s^l$  and  $b^l$  are the limit sell price and the limit buy price, respectively. Within each region, the MB shares are graphed as a function of the terminal value  $v$ . The heavy, step function is the optimal number of MB shares as determined by the critical values  $v_0, v_1, v_2, \dots, v_4, \dots, v_8, v_9$  on the  $x$  axis.  $\mu$  is the unconditional mean,  $a$  is the ask price, and  $v_H$  is the upper bound of  $v$ .

Region I is  $v < \$45.27$ , Region II is  $\$45.27 < v < \$47.63$ , and Region III is  $v > \$47.63$ . The optimal strategy is

- in Region I,
  - a combined MB for zero to four shares and a LS with  $s^l = \$45.50$  and
  - an isolated LB with  $b^l = \$44.50$ ;
- in Region II,
  - a combined MB for seven to ten shares and a LS with  $s^l = \$47.63$  and
  - an isolated LB with  $b^l = \$44.50$ ;
- in Region III,
  - a MB only for ten shares, no LS, and
  - an isolated LB with  $b^l = \$44.50$ .

Within each region, the MB quantity is graphed as a function of the terminal value  $v$ . The optimal value of  $MB^l$  is a step function determined by the critical values shown on the  $x$ -axis. For example, if the terminal value  $v \in (v_2, v_3]$ , then the optimal  $MB^l = 3$ .

Figure 4 also illustrates the main points of Propositions 1 and 2. With regard to Proposition 1, the step function for the informed trader's strategy is positive in Regions I and II (which are both inside the spread  $v < a$ ). Hence we see that the informed trader optimally submits a positive MB for values of  $v$  inside the spread. With regard to Proposition 2, Fig. 4 illustrates that a combined MB-LS (rather than a MB only) is optimal in Regions I and II.

#### 4. CONCLUSION

By adding limit orders to a standard model of market microstructure, we demonstrate the complex structure of the optimal strategy that results. The optimal strategy includes

- submitting a market order inside the bid-ask spread,
- a combined MB-LS submission, where the component orders may cross with each other, and the limit order acts as a safety net for the market order in the opposite direction, and
- different limit sell prices in different regions.

In many respects our model only scratches the surface of the complexities introduced by limit orders. Future research could probe deeper by generalizing our model. For example, one extension would be a multiperiod version. Unexecuted limit orders would be carried over to the next period and the market maker would make an updated inference about the value of the security before issuing new bid-ask quotes. A second extension would allow the market makers to act strategically, as in Rock (forthcoming). A third extension would endogenize the uninformed orders.

#### APPENDIX A

*Proof of Lemma 1.* (1) A limit order with a limit sell price  $s^l > a$  would never be executed. (2) From Eq. (2), it is clear that a limit sell price  $s^l < v$  would yield negative expected profits. (3) Reducing the limit sell price  $s^l$  reduces the profit per share conditional on execution  $s^l - v$ , but increases the chances of execution by increasing the chance that the informed limit sell price will beat out the uninformed limit sell price,  $s^l <$

$s^U$ . Once  $s^I$  is reduced down to  $\mu$ , then the informed limit sell price always beats out the uninformed limit sell price  $s^U \in [\mu, a]$ , and the informed trader would be strictly worse off if  $s^I$  were lowered any further. (4), (5), and (6) are analogous to (1), (2), and (3). Q.E.D.

*Proof of Lemma 2.* (1) The expected price of the market buy is a weighted combination of five prices corresponding to the five events listed. From the distribution assumptions and from Lemma 1, each of the five prices is in  $[\mu, a]$  and there is a nonzero weight on both endpoints, hence the weighted average is in  $(\mu, a)$ . (2) is analogous to (1). Q.E.D.

*Proof of Proposition 1.* From Lemma 2, we know that  $E[p_{MB}] < a$ , for any  $MB \geq 1$ , and hence there exist some values of  $v$  such that  $a > v > E[\tilde{p}_{MB}]$ . For these values of  $v$ , combined with a limit sell for any number of limit sell shares  $N_{LS}^I$  and any limit sell price  $s^I$ , it is more profitable to submit a market buy for a positive number of shares than for zero shares (i.e.,  $MB^I(v - E[\tilde{p}_{MB}]) > 0$ ). The proof for the other side is analogous. Q.E.D.

*Proof of Lemma 3.* (1) From Lemma 1, it is clear that when  $v > a$ , then  $v - s^I$  is strictly positive. Hence, a nonzero limit sell order would yield negative expected profits. (2) is analogous to (1). (3) From Lemma 2, it is clear that when  $v > \mu$ , then  $v - E[\tilde{p}_{MS}]$  is strictly positive. Hence, a nonzero market sell order would yield negative expected profits. (4) is analogous to (3). Q.E.D.

*Proof of Lemma 4.* (1) If the informed LS trades against an uninformed MB, then it generates a positive profit per share and more shares yield more total profit. If the informed LS trades against the informed MB, then it reduces the MB loss per share and more shares yield more total loss reduction. Hence more LS shares are always better. (2) is analogous to (1). Q.E.D.

*Proof of Proposition 2.* Both follow immediately from Lemma 4. Q.E.D.

*Assumptions for Proposition 3.* We assume that

- $v$  has a continuous uniform distribution over  $[v_L, v_H]$ ,
- $N_{LS}^U = \begin{cases} -Q & \text{with prob.} = \pi_{LS}, \\ 0 & \text{with prob.} = 1 - \pi_{LS}, \end{cases}$
- $N_{LB}^U = \begin{cases} Q & \text{with prob.} = \pi_{LB}, \\ 0 & \text{with prob.} = 1 - \pi_{LB}, \end{cases}$
- $s^U = h$ , where  $h$  is a constant in the interval  $(\mu, a)$ ,

- $b^U = l$ , where  $l$  is a constant in the interval  $(b, \mu)$ ,
- $M^U$  is a discrete random variable defined on the integers  $\{-Q, \dots, -1, 0, 1, \dots, Q\}$ ,

- $MB^U = \begin{cases} M^U, & \text{when } M^U > 0, \\ 0 & \text{otherwise,} \end{cases}$
- $MS^U = \begin{cases} M^U, & \text{when } M^U < 0, \\ 0 & \text{otherwise.} \end{cases}$

$M^U$  can be thought of as the net market orders from uninformed traders.<sup>22</sup> We do not make a specific distributional assumption about  $M^U$ .

*Proof of Proposition 3.* Multiplying by  $N$  and conditioning on the number of shares of the informed trader's market buy ( $MB^I = k$  shares, where  $k$  can equal zero) under the optimal strategy, Eq. (4) in Section 2.4 can be written as

$$\sum_{k=0}^Q \Pr(MB^I = k) E[X | MB^I = k] (E[v | MB^I = k] - a) = 0, \quad (5)$$

where  $\Pr(MB^I = k)$  is the probability that  $MB^I = k$ ,  $E[X | MB^I = k]$  is the expected number of market maker sells conditional on  $MB^I = k$ , and  $E[v | MB^I = k]$  is the expected value of  $v$  conditional on  $MB^I = k$ .

The assumption that  $v$  is uniformly distributed leads to simple expressions. The probability that  $v$  is in a specific interval  $[v_{k-1}, v_k]$  is  $(v_k - v_{k-1}) / (v_H - v_L)$  and the expected value of  $v$  in that interval is  $(v_k + v_{k-1}) / 2$ . Substituting in (5), the market maker's zero expected profit condition becomes

$$\sum_{k=0}^Q \left( \frac{v_k - v_{k-1}}{v_H - v_L} \right) E[X | MB^I = k] \left( \frac{v_k + v_{k-1}}{2} - a \right) = 0.$$

The uniform distribution also simplifies the equation for the  $k$ th critical value  $v_k$  into a linear function of  $a$ . Let this linear function be denoted  $v_k = \alpha_k a + \beta_k$  for  $k \in \{0, \dots, Q-1\}$ , where expressions for  $\alpha_k$  and  $\beta_k$  are given in Appendix B. Multiplying both sides by  $2(v_H - v_L)$ , rearranging, and

<sup>22</sup> In effect we have dropped the independence assumption between  $MB^U$  and  $MS^U$ . This is done just as a matter of computer programming convenience in order to reduce the number of combined outcomes from these two random variables from  $11 \times 11 = 121$  outcomes to  $11 + 10 = 21$  outcomes. There is no conceptual difficulty in restoring the independence assumption.

substituting the linear expression for the  $k$ th critical value, we obtain a quadratic equation for  $a$ ,

$$\sum_{k=0}^Q (\alpha_k a + \beta_k - \alpha_{k-1} a - \beta_{k-1}) (\alpha_k a + \beta_k + \alpha_{k-1} a + \beta_{k-1} - 2a) E[X | MB^1 = k] = 0.$$

This leads immediately to the solution given by the quadratic formula. Q.E.D.

APPENDIX B

This appendix provides additional detail on the analytic solution given in Proposition 3. The plan of the appendix is to:

- provide a more explicit version of the equation for the critical value  $v_k = \alpha_k a + \beta_k$ ,
- substitute for  $E[p_k]$  and  $s_k^{1*}$  functional forms which are linear in the ask price  $a$ ,
- solve for  $\alpha_k$  and  $\beta_k$ , which substitute directly into the analytic solution, and
- provide additional details on how to calculate the informed trader's expected price of a MB ( $E[p_k]$ ) and the informed trader's expected number of shares that will execute under a LS ( $E[LS^1]$ ).

The critical value  $v_k$  is determined by where the informed trader is indifferent between trading the combination  $MB^1 = k$  shares and the corresponding optimal limit sell price  $s_k^{1*}$  versus an alternative combination of  $MB^1 = j$  shares (where  $j > k$ ) and the corresponding optimal limit sell price  $s_j^{1*}$ . Referring to Eq. (3) in Section 2.3 and substituting  $J_{MB}(\cdot)$  from Eq. (2) for the profit function on the MB side, we obtain

$$k(v_k - E[p_k]) + E[LS_k^1](v_k - s_k^{1*}) = j(v_k - E[p_j]) + E[LS_j^1](v_k - s_j^{1*}),$$

where  $j > k$  is the profit maximizing combination. With all of these critical values, when a particular terminal value  $v$  is in the interval  $[v_{k-1}, v_k]$ , then it is optimal for the informed trader to submit a MB for  $k$  shares combined with a LS at the optimal LS price  $s_k^{1*}$ .

Because of the uniform distribution of  $v$  under the special distribution assumptions, the expected price of a market buy can be represented in a simple linear form,  $E[p_k] = f_k a + g_k$ . Similarly, given that  $s^U = h$ , then

$s_k^{1*}$  is either  $h$  or  $a$ . It simplifies matters a great deal to represent these two values as linear functions of the ask price  $a$ ,  $s_k^{1*} = c_k a + d_k$ . Thus, when  $s_k^{1*} = a$ , then  $c_k = 1$  and  $d_k = 0$ , and when  $s_k^{1*} = h$ , then  $c_k = 0$  and  $d_k = h$ . Substituting these linear forms in the equation above, we get

$$k(v_k - E[f_k a + g_k]) + E[LS_k^1](v_k - [c_k a + d_k]) = j(v_k - E[f_j a + g_j]) + E[LS_j^1](v_k - [c_j a + d_j]).$$

Solving for the critical value  $v_k$ , we obtain

$$v_k = \alpha_k a + \beta_k,$$

where

$$\alpha_k = \frac{k f_k - j f_j + c_k E[LS_k^1] - c_j E[LS_j^1]}{k - j + E[LS_k^1] - E[LS_j^1]},$$

and

$$\beta_k = \frac{k g_k - j g_j + d_k E[LS_k^1] - d_j E[LS_j^1]}{k - j + E[LS_k^1] - E[LS_j^1]}.$$

To complete the set of critical values on the MB side, we add the highest and lowest bounds,  $v_{-1} = v_L$  and  $v_Q = v_H$ . These critical values imply the coefficients  $\alpha_{-1} = 0$ ,  $\beta_{-1} = v_L$ ,  $\alpha_Q = 0$ , and  $\beta_Q = v_H$ . All of these coefficients for  $\alpha_k$  and  $\beta_k$  substitute into the analytic solution in Section 2.5 in order to determine the ask price  $a$ .

To compute the formulas above, we need to calculate  $f_k$ ,  $g_k$ , and  $E[LS_k^1]$  (for both  $S^1 = h$  and  $S^1 = a$ ), and  $f_j$ ,  $g_j$ , and  $E[LS_j^1]$  (for both  $S^1 = h$  and  $S^1 = a$ ), for all  $j > k$ . In order to do this, we make the following definitions:

- $NMB^1 \equiv$  the net informed MB  $\equiv MB^1 - MS^1 \equiv$  the informed MB less the uninformed MS (i.e., net of the MS that the MB crosses with);
- $MB^U \leq Q - NMB^1$  is the "slack" state in which the uninformed MB is less than the informed LS shares less NMB;
- $F_{MB} \equiv$  the cumulative probability of being in the "slack" state;
- $x_k \equiv$  the expected fraction of shares of a MB for  $k$  shares that do not cross with MS shares, in other words, the fraction  $1 - x_k$  does cross with MS shares;
- $w_k \equiv$  the fraction of shares of a MB for  $k$  shares crossing with a LS

TABLE III  
INFORMED TRADER'S  $E[LS^i]$  AND  $E[p_{MB}]$  BY TWO LIMIT SELL PRICES UNDER FOUR STATES

State conditions	Probability	A: For $s^1 = h$	
		Informed trader's $E[LS^i   \text{state}]$	Informed trader's $E[p_{MB}   \text{state}]$
$N_{LS}^U = 0, MB^U \leq Q - NMB^1$	$(1 - \pi_{LS})F_{MB}$	$-(E[MB^U   MB^U \leq Q - NMB^1] + NMB^1)$	$x_k h + (1 - x_k)\mu$
$N_{LS}^U = 0, MB^U > Q - NMB^1$	$(1 - \pi_{LS})(1 - F_{MB})$	$-Q$	$x_k[w_k h + (1 - w_k)a] + (1 - x_k)\mu$
$N_{LS}^U = -Q, MB^U \leq Q - NMB^1$	$\pi_{LS} F_{MB}$	$-(E[MB^U   MB^U \leq Q - NMB^1] + NMB^1)$	$x_k h + (1 - x_k)\mu$
$N_{LS}^U = -Q, MB^U > Q - NMB^1$	$\pi_{LS}(1 - F_{MB})$	$-Q$	$x_k h + (1 - x_k)\mu$

State conditions	Probability	B: For $s^1 = a$	
		Informed trader's $E[LS^i   \text{state}]$	Informed trader's $E[p_{MB}   \text{state}]$
$N_{LS}^U = 0, MB^U \leq Q - NMB^1$	$(1 - \pi_{LS})F_{MB}$	$-(E[MB^U   MB^U \leq Q - NMB^1] + NMB^1)$	$x_k a + (1 - x_k)\mu$
$N_{LS}^U = 0, MB^U > Q - NMB^1$	$(1 - \pi_{LS})(1 - F_{MB})$	$-Q$	$x_k a + (1 - x_k)\mu$
$N_{LS}^U = -Q, MB^U \leq Q - NMB^1$	$\pi_{LS} F_{MB}$	0	$x_k h + (1 - x_k)\mu$
$N_{LS}^U = -Q, MB^U > Q - NMB^1$	$\pi_{LS}(1 - F_{MB})$	$-(E[MB^U   MB^U > Q - NMB^1] + NMB^1 - Q)$	$x_k[w_k h + (1 - w_k)a] + (1 - x_k)\mu$

Note.  $LS^i$  is the realized number of shares traded by a LS,  $p_{MB}$  is the price of a MB,  $s^1$  is the limit sell price,  $h$  is the uninformed limit sell price in this example.  $N_{LS}^U$  is the number of shares requested by an uninformed LS,  $MB^U$  is an uninformed MB,  $Q$  is the maximum number of shares that can be bought or sold,  $NMB^1$  is the net informed MB,  $\pi_{LS}$  is the probability that an uninformed LS is submitted,  $F_{MB}$  is the probability of the "slack" state ( $MB^U \leq Q - NMB^1$ ),  $x_k$  is the expected fraction of MB for  $k$  shares that do not cross with MS,  $\mu$  is the unconditional mean,  $w_k$  is the expected fraction of MB for  $k$  shares crossing with a LS at the lowest LS price, and  $a$  is the ask price.

at the lowest LS price available, in other words, the fraction  $1 - w_k$  is the overflow that goes to the next LS or crosses with the market maker.

Table III lists an expression for each of these variables under four different states. Panel A does this for an informed LS with  $s^1 = h$ . Panel B repeats the exercise for an informed LS with  $s^1 = a$ .

To illustrate how to read Table III, we will explain the first row in Panel A for state 1. Panel A is for the case in which there is an informed LS with  $s^1 = h$  requesting  $N_{LS}^U = -Q$  shares. Reading across the first row for state 1, the state conditions are that there is no uninformed LS ( $N_{LS}^U = 0$ ) and the uninformed MB is in the "slack" state (see above). The corresponding probabilities of these state conditions are  $1 - \pi_{LS}$  and  $F_{MB}$ . The informed trader's expected number of shares traded via the LS conditional on being in state 1 is the expected number of shares conditional on being in the "slack" state ( $E[MB^U | MB^U \leq Q - NMB^1]$ ) plus the net informed MB ( $NMB^1$ ). The informed trader's expected price for a MB conditional on being in state 1 is the weighted average of the shares that cross with the informed LS at  $h$  and the shares that cross with the uninformed MS at  $\mu$ , with the weights being  $x_k$  and  $1 - x_k$ , respectively. The rest of the rows for the other states can be read similarly.

Each of the rows in Panel A in Table III assumes an informed MB of  $k$  shares. In order to calculate the coefficients of  $E[p_k] = f_k a + g_k$  and  $E[LS_k^i]$ , we simply need to calculate the expectation. We do this by (1) taking a probability-weighted sum over all possible realizations of the net market orders from the uninformed ( $M^U$ ) within a state and then (2) take a probability-weighted sum over all four states.

The results are

$$f_k = \sum_{i=k-1}^0 \Pr(M^U = i) \left( \frac{k+i}{k} \right) [(1 - \pi_{LS})(1 - F_{k+i})(1 - w_{k+i})] + \Pr(M^U > 0)[(1 - \pi_{LS})(1 - F_k)(1 - w_k)],$$

$$g_k = \left\{ \begin{array}{l} \sum_{i=k-1}^0 \Pr(M^U = i) \left( \frac{k+i}{k} \right) \\ [F_{k+i} + (1 - F_{k+i})(\pi_{LS} + (1 - \pi_{LS})(1 - w_{k+i}))] \\ + \Pr(M^U > 0)[F_k + (1 - F_k)(\pi_{LS} + (1 - \pi_{LS})(1 - w_k))] \end{array} \right\} h$$

$$+ \left( 1 - \sum_{i=k-1}^0 \Pr(M^U = i) \left( \frac{k+i}{k} \right) - \Pr(M^U > 0) \right) \mu,$$

$$E[LS_k^i] = \sum_{i=k-1}^0 \Pr(M^U = i) \left( \frac{k+i}{k} \right)$$

$$\begin{aligned}
& \{F_{k+i}\{-E[MB^U] | MB^U \leq Q - NMB^U] + k + i\} \\
& + (1 - F_{k+i})\{-Q\} + \Pr(M^U > 0) \\
& \{F_k\{-E[MB^U] | MB^U \leq Q - NMB^U] + k\} + (1 - F_k)\{-Q\},
\end{aligned}$$

where

$$x_k = \sum_{i=k+1}^0 \Pr(M^U = i) \left( \frac{k+i}{k} \right) + \Pr(M^U > 0),$$

and

$$w_k = E \left[ \frac{Q}{k + M^U} \middle| MB^U \leq Q - NMB^U \right].$$

Using Table III, it is straightforward to write out the analogous expressions for  $f_j$ ,  $g_j$ ,  $E[LS_k^j]$ , when  $s^j = a$ ,  $E[LS_k^j]$  when  $s^j = h$  and  $E[LS_k^j]$  when  $s^j = a$ .

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