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Pricing Convertible Bonds with Interest Rate, Equity, Credit and FX Risk

ISMA Centre Discussion Papers In Finance 2001-14

First Draft: 30th November 2001

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Abstract

Convertible bonds are hybrid securities whose pricing relies on a set of complex inter-dependencies due to the sensitivity to interest rate risk, underlying (equity) risk, FX risk, and credit risk, and due to the convertible bond's early exercise American feature. We present a two factor model of interest rate and equity risk that is implemented using the Crank-Nicholson technique on the discretized pricing equation with projective successive over-relaxation. This paper extends a methodology proposed in the literature (TF[98]) to deal with credit risk in a self-consistent way, and proposes a new methodology to deal with FX sensitive cross-currency convertibles. A technique for extracting the price of vanilla options struck on a synthetic asset, the foreign equity in domestic currency, is employed to obtain the implied volatility for these options. These implied volatilities are then used to obtain the local volatility for use in the numerical routine. The model is designed to deal with most of the usual contractual features such as coupons, dividends, continuous and/or Bermudan call and put clauses. We suggest that credit spread adjustments in the boundary conditions can be made, to account for the negative correlation between spreads and equity. Detailed description of the numerical methods and the discretization schemes, together with their accuracy, are provided.

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JEL Classification: C63, C13, C15

Keywords: cross-currency convertibles, credit spread, interest rate risk, American feature, local volatility, Crank-Nicholson.

I would like to thank my supervisor Prof. Carol Alexander, and Dr. Jacques Pezier for their valuable comments and suggestions. I would also like to thank Steffen Hennig (Deutsche Bank AG) for many stimulating discussions on this subject, and for joint computational work. All errors are my responsibility.

1. Introduction

Convertible bonds are hybrid securities which confer upon the owner the right to receive a fixed income stream during the life of the convertible with the embedded right to forego the fixed income stream and convert into a prescribed amount of underlying equity any time during the life of the instrument[1]. Because convertibles will be exercised when the issuer does well, convertible debt acts to lower the issuer's cost of debt financing through the implicit call option on the issuer's stock that is implicit in the convertible. In essence, the convertible bond is a mixture of a corporate bond and a warrant on the underlying equity. The decision to exercise the warrant terminates the bond component of the instrument. Its corporate debt and equity nature renders it sensitive to equity, interest rate, and credit risk. Furthermore, the special case of FX risk arises for those convertibles that pay coupons and face value in some foreign currency but convert into some domestic equity.

The worldwide convertibles market by issue size is in excess of US$ 400 billion and highly developed markets for convertibles exist in the United States, Japan, England, France, Australia, Canada, Sweden, and Switzerland. The majority of issues in these markets consist of domestic convertibles denominated in a local currency and convertible into domestic equity. Details of outstanding issue size, maturity, initial premium[2] for the US market case are provided in Appendix 2.

In addition to domestic convertible markets, there is a sizeable Eurobond convertible market in which currency sensitive convertibles are actively issued and traded.[3] This market as a whole grew from US$ 8.1 billion in 1990 to US$ 14 billion in 1995 (C[98]). The Eurobond convertible market contains, as a subset, convertible issues that convert into the issuer's (domestic) equity but are denominated (i.e. pay coupons and face value) in some non-domestic currency, such as USD or EUR. These instruments are naturally sensitive to FX risk.

One of the largest convertible bond markets, by issue size, is Japan which as of 1998 had a 40% share of the world market in convertibles (C[98]). The Japanese market also produced one of the largest ever issues of FX-sensitive convertibles. In September 1995, the Mitsubishi Bank issued a US$2 billion resetable, mandatory, exchangeable convertible security, maturing

[1] In some cases the convertible may only be converted during a prescribed time period during its life.
[2] Defined as the percentage difference between the conversion price of equity and the issue price.
[3] Eurobond convertible refers to the broad market category encompassing all convertibles issued outside the issuer's domestic market. The denomination currency can be issuer's domestic currency (no FX risk) or some foreign currency (leading to FX risk).

Nov. 2002, mandatorily converting into Japanese equity (ADS) at maturity. The mandatory conversion was sweetened with an upward resetting conversion ratio over time should the *dollar* value of the Japanese equity depreciate below a certain level on fixed reset dates. Clearly, a depreciating Japanese Yen would decrease the dollar value of the Japanese equity, and thereby impact the value of the convertible. The details of this instrument are provided in Appendix 3.

A rapidly growing continental Eurobond convertible market is the Swiss market, which is unique in that the majority of companies that issue convertible debt in Swiss francs are not domestic firms[4]. Examples of outstanding convertible issues on the Swiss markets include the USD denominated Hutchison Whampoa International 2% coupon, convertible into sterling denominated shares in Jan. 2004. Thus the Swiss market provides another incentive for the inclusion of FX risk in a convertible pricing model.

Additional features complicate the pricing problem. These include the American early exercise feature, time dependent call by the issuer and (Bermudan or continuous) put by the investor, resetting conversion ratios. Further complexity is caused by the hybrid credit risk nature of convertibles. The convertibility feature of the convertible bond poses no credit risk to the investor. This is because the issuer can always deliver its own equity[5] while the fixed income characteristics such as interest payments, the face value repayment, and the put payment associated with the put event is subject to credit risk as the issuer needs cash to fulfil its payment obligations (see TF[98]).

It is possible to view the credit spread associated with the same issuer's nonconvertible debt as an upper bound for the credit risk associated with the convertible bond, a fact that is often used by practitioners for valuing convertibles with a rule of thumb approach. Using this view, a trader might use, say, 60% of the credit spread for nonconvertible debt as the appropriate discount rate for the convertible bond in the pricing equation. The pitfalls of using such ad-hoc approaches is discussed in TF[98], who suggest a one factor coupled PDE approach consistent with Black-Scholes to deal with credit risk. This paper extends their model and includes interest rate risk in the coupled pricing PDE.

[4] C[98] reports that approximately 90% of issuers are foreign firms.

[5] Unless the convertible is exchangeable into the equity of a firm other than the issuer, in which case the issuing firm has to purchase the equity with cash.

We propose a change of numeraire technique to account for the FX risk in cross-currency convertible bonds, without increasing the dimensionality of the problem. It is interesting that the pricing of FX-sensitive convertibles has altogether not attracted a high degree of interest in the literature to date. The model we develop for international convertibles has the property that domestic convertibles are a subset and can easily be accommodated in the same framework.

This paper presents a two factor (equity and interest rate) parabolic *coupled* PDE approach with change of numeraire to price international convertibles with credit risk. The credit risk of convertibles is dealt with by introducing an auxiliary asset called the cash only part of the convertible (COCB), which pays state contingent fixed interest payments only, and extends the approach introduced by TS[98], which also satisfies a similar pricing equation with its boundary and initial conditions coupled to the convertible security. The two pricing equations are solved simultaneously due to this coupling. As in the basic TF[98] model, we assume that the capital structure of the issuer consists of senior corporate debt, with associated credit spread over treasury returns, and the junior convertible issue. This assumption in the model can be relaxed without difficulty, as discussed in TF[98].

Furthermore, we propose to use the local volatility approach in lieu of constant equity volatility, within the change of numeraire framework. To our knowledge, this approach has not been used in the convertible pricing literature to date, and although it is mentioned, in a footnote, as a possibility in a recent paper (HNB[01]), the suggestion relates to the domestic convertible bond problem.

The change of numeraire approach relies on the stochastic process followed by the foreign equity in domestic currency. As a result, there is no market-given volatility smile for options struck on this synthetic process. We propose to back out the price of options struck on the process followed by the foreign equity in domestic currency, using the implied volatility for vanilla FX and foreign equity (in foreign currency) options. This is accomplished numerically by evaluating an expectation over the joint density of the FX rate and the foreign equity processes, over all possible (combinations of) strikes for FX and foreign equity options. The prices so obtained are used to back out the implied volatility for the foreign equity in domestic currency. These implied volatilities are then used to obtain local volatility for use in the numerical routine.

An issue plaguing the TF[98] approach, which is extended in this paper, is the correlation between credit spreads and equity, which is unaccounted for in the TF[98] framework. Credit spreads are sensitive to firm value, and will tend to increase when equity prices decrease, and vice versa. TF[98] use a constant credit spread in their paper which is used to discount the COCB. By their construction, this spread is invariant to changes in the credit quality of the issuer whatever the value of the issuer's equity in the future. Our framework can be used to adjust the credit spread for low equity values in the boundary conditions of the pricing PDE. To this end we provide a full description of the pricing PDE in all the regions of the grid. On the other hand, the credit spread for high values of equity will in effect drop out of the pricing PDE due to the fact that the COCB will be valued at zero in high equity regions in the grid.

A Crank-Nicholson scheme, with adaptive grid to match time steps to jump events, such as coupon payments, is employed. The American early exercise feature is posed as a linear complementarity problem and is tackled using the projective successive over-relaxation (PSOR) method. This method has some shortcomings, as detailed in the literature (see ZS[99] and WK[97], and section 2 of this paper) in that it cannot give the exact location of the free boundary at which it is optimal to convert the convertible bonds. By employing smaller equity steps in our grid the adverse effect of the PSOR method is reduced: the PSOR scheme requires a diagonally dominant propagator matrix for convergence, which is enforced using small time steps, and the high number of equity steps reduces the inaccuracy due to projection.

We also provide details on the pricing PDE and its discretization in various regions of the grid in relation to its boundary/initial conditions. This serves two purposes. First, this provides a scope to investigate the stability of the discretization scheme and to shed light on the implementation details. The second purpose is to provide a complete description of the grid used: this is where credit spread adjustments in the boundary conditions can be made, due to the negative correlation between spreads and equity.

The paper is organised as follows. Section 2 provides a brief critical review of literature on pricing convertibles. Section 3 details the basic model for FX, equity, and interest rate; it provides details of the change of numeraire technique, the local volatility technique used, and the credit risk approach in addition to the pricing equations and boundary and initial conditions on the grid. Section 4 provides the numerical details of the model. Section 5 concludes. A detailed exposition on the numerical techniques, the discretization, and accuracy of the schemes are provided in the appendix.

2. Literature Review

The literature on numerical methods for solving the convertible bond problem in the BS[73] framework originates in BSh[77] and I[77]. They use firm value models for domestic convertibles without credit risk, using only one factor. BSh[77] use the explicit finite difference scheme to solve the problem numerically. BSh[80] then extend their earlier model to include interest rate risk.

All three earlier papers neglect to mention the American early exercise feature for convertibles, nor do they allow for credit risk, volatility calibration or FX risk. WDH[93] show that the American feature of convertible bonds can be interpreted as a linear complementarity problem, which, after discretization, can be solved using a so-called projection version of the successive over-relaxation technique from numerical analysis. This projected successive over-relaxation (PSOR) technique avoids solving for the early exercise boundary. WK[97] use a transformation method to solve explicitly for the early exercise boundary for an American put with dividends, using a one factor PDE. Consequently ZS[99] extend the WK[97] technique to two factors, and study convertible bonds. Their paper neglects to incorporate the call and put features of convertibles, each of which constitutes a further coupled non-linear PDE, complicating the numerical implementation enormously.

TF[98] are the first to model credit risk explicitly in a Black-Scholes world. They solve for the convertible bond value using a one factor coupled PDE model with the explicit finite difference technique. DL[99] propose a "two and a half factor" lattice model with stochastic interest rate, equity, and hazard rate which calibrates to the initial term structure via a HW[90] interest rate framework and to the term structure of credit spreads. The DL[99] framework does not address FX risk and uses a constant equity volatility assumption. Recently HN[01] present a one factor model to price Japanese resetting convertibles, and propose a method for accounting for the call notice period and the reset feature without increasing the dimension of the problem.

3. The Model

3.1 Model Basics

We can set up a market consisting of a foreign asset paying continuous dividend yield γ, a local risk-free bond, a foreign risk-free bond, where the (correlated) assets follow the following processes in the objective P-measure: [6]

$$\begin{aligned} dS_f &= S_f(\alpha_f - \gamma)dt + S_f \sigma_f(S_f, t)d\overline{W} \\ dB_d &= r_d B_d dt \\ dB_f &= r_f B_f dt \end{aligned} \tag{1}$$

Here, $d\overline{W}$ is a domestic (multidimensional) Brownian motion under the objective measure P, and $\sigma_f(X,t)$ is some local volatility function depending on the volatility smile for foreign equity (vanilla) options.

We choose the following one factor CIR[85] model to characterise domestic interest rate dynamics, although the pricing framework is general and can accommodate other interest rate models, including those which calibrate to the initial term structure, such as HW[90].

$$dr_d(t) = (\theta - ar_d(t)dt + w\sqrt{r_d(t)}dW_r \tag{2}$$

It is proposed that the exchange rate X, the number of foreign units of currency per one unit of domestic currency, follows the geometric Brownian motion

$$dX = X\alpha_X dt + X\sigma_X(X, t)d\overline{W}$$

where $\sigma_X(X,t)$ is local volatility function depending on the volatility smile for FX vanilla options.

Volatility is functionally dependent on the underlying and time and hence, contingent upon the realized value of the underlying in the future, is fully determined (see R[99]). So let us

[6] Identical notation to Bjork(1998) is used for the basic framework.

write σ_X and σ_f as shorthand for the *local* volatilities from here on, suppressing dependence on underlying and strike for notational convenience. We note that:

$$\overline{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \text{ and } \sigma = \begin{bmatrix} \sigma_X \\ \sigma_f \end{bmatrix} = \begin{bmatrix} \sigma_{X1} & \sigma_{X2} \\ \sigma_{f1} & \sigma_{f2} \end{bmatrix} \tag{3}$$

where σ_X and σ_f are 1x2 vectors in the above definition (3). In particular, correlations are expressed in terms of a vector product,

$$\rho_{f,X} = \frac{\sum_{i=1}^{2} \sigma_{fi}\sigma_{Xi}}{\|\sigma_f\|\|\sigma_X\|} = \frac{\sigma_f \sigma_X^T}{\|\sigma_f\|\|\sigma_X\|}$$

As the market with local volatility is complete (Rebonato, 1999), we can use arbitrage arguments to derive the processes for the FX rate, and the foreign equity in domestic currency, under the domestic risk neutral measure, Q.

It is easy to show[7] that the foreign exchange rate follows the geometric Brownian motion with local rate of return $r_d - r_f$ in the Q measure:

$$dX = X(r_d - r_f)dt + X\sigma_X dW$$

where dW is a multidimensional Brownian motion in the domestic Q measure.

3.2 Measure change

To derive the domestic price of the foreign equity, denoted $\widetilde{S}_f = X(t)S_f$, Itô's lemma is invoked:

[7] Musiela and Rutkowski (1998), pp. 161

$$d(XS_f) = d\widetilde{S}_f = \frac{\partial \widetilde{S}_f}{\partial X} dX + \frac{\partial \widetilde{S}_f}{\partial S_f} dS_f + \frac{\partial^2 \widetilde{S}_f}{\partial X \partial S_f} dX dS_f$$
$$= S_f (X\alpha_X dt + X\sigma_X d\overline{W}) + X(S_f(\alpha_f - \gamma)dt + S_f \sigma_f d\overline{W}) + XS_f(\sigma_{X1}\sigma_{f1} + \sigma_{X1}\sigma_{f1} + \sigma_{X1}\sigma_{f1})dt$$
$$= XS_f(\alpha_X + \alpha_f - \gamma + \sigma_f \sigma_X)dt + XS_f(\sigma_f + \sigma_X)d\overline{W}$$
$$= \widetilde{S}_f(\alpha_X + \alpha_f - \gamma + \sigma_f \sigma_X^T)dt + \widetilde{S}_f(\sigma_f + \sigma_X)d\overline{W}$$

For $\widetilde{S}_f$ to have the correct risk neutral drift of $r_d - \gamma$, introduce the measure transformation:

$$(\sigma_f + \sigma_X)d\overline{W} = (\sigma_f + \sigma_X)dW + r_d - (\alpha_X + \alpha_f + \sigma_f \sigma_X^T)dt \tag{4}$$
$$\Rightarrow d\widetilde{S}_f = \widetilde{S}_f(r_d - \gamma)dt + \widetilde{S}_f(\sigma_f + \sigma_X) \text{ where } dW \text{ is the domestic risk neutral measure}$$

Here dW is the domestic (multidimensional) Brownian motion associated with the risk neutral probability measure.

Furthermore we assume that the interest rate SDE and the SDE for $\widetilde{S}_f$ are correlated through

$$d\widetilde{S}_f dr = \widetilde{S}_f(\sigma_f + \sigma_X) w\sqrt{r}\rho_{\widetilde{S},r} dt \,.$$

3.3 Risk neutrality and the basic pricing equation

Itô's lemma applied to the convertible bond $u(\widetilde{S}_f, r, t)$ which depends on the foreign equity in domestic currency, the domestic interest rate r, and time yields the SDE followed by the differential process *du*:

$$\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r}(\theta - ar) + \frac{\partial u}{\partial \widetilde{S}_f}(r - \gamma)\widetilde{S}_f + \frac{1}{2}(\sigma_X^2 + 2\rho_{f,X}\sigma_X\sigma_f + \sigma_f^2)\widetilde{S}_f^2 \frac{\partial^2 u}{\partial \widetilde{S}_f^2} \right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 u}{\partial r^2} w^2 r + \frac{\partial^2 u}{\partial \widetilde{S}_f \partial r} \widetilde{S}_f(\sigma_X + \sigma_f) w\sqrt{r}\rho_{\widetilde{S},r} \right) dt + \frac{\partial u}{\partial r} w\sqrt{r} dW_r$$
$$+ \frac{\partial u}{\partial \widetilde{S}_f} \widetilde{S}_f(\sigma_X + \sigma_f) dW = du(\widetilde{S}_f, r, t)$$

Risk-neutrality arguments then lead to the pricing equation for the convertible bond which pays domestic interest but converts into foreign equity

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}(\theta - ar)+\frac{\partial u}{\partial \widetilde{S}_f}(r-\gamma)\widetilde{S}_f+\frac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\widetilde{S}_f^2\frac{\partial^2 u}{\partial \widetilde{S}_f^2}+\frac{1}{2}\frac{\partial^2 u}{\partial r^2}w^2 r$$
$$+\frac{\partial^2 u}{\partial \widetilde{S}_f \partial r}\widetilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\widetilde{S},r}-ru=0 \tag{5}$$

This equation is valid only up to the early exercise boundary, at which $u(\widetilde{S}_f, r, t) = CR.\widetilde{S}_f$ (provided the conversion price is below the call price plus accrued interest). Here, CR refers to the conversion ratio.

It is assumed that the convertible bondholder (the "domestic investor") converts only when $CR.\widetilde{S}_f$ is high enough compared to:

$V_L(t) = \max(PP(t) + \text{AcInt}(t), \text{Bond\_Value}(t))$

where $PP(t)$ is the prevailing put price at time t (a step function of time), AcInt(t) is the accrued interest at time t since the last coupon payment, and Bond_Value(t) is the time t present value of the fixed income flows from the convertible bond. Thus $V_L(t)$ constitutes a lower value for the convertible.

The call value of the convertible is assumed to be resetting (and is cast as a step function), possibly with hard call protection[8], and can be cast as an upper value for the convertible:

$V_U(t) = V_{call}(\widetilde{S}_f, r(t), t)$

Equivalently this is an upper value for $\widetilde{S}_f$, thereby constituting an upper price bound for the instrument on the finite difference grid.

3.4 Local Volatility

It is supposed that prices for OTC call/put options struck on S_f and X are available up to the maturity time of the convertible bond[9], and hence their implied volatilities are also available up to the maturity of the convertible bond. Then the local volatility $\Phi(\widetilde{S}_f(t),t)$ for vanilla options struck on $\widetilde{S}_f$ can be found numerically from the implied volatility for $\widetilde{S}_f$. This local volatility form is supplied to the convertible bond PDE in lieu of constant volatility.

The problem is that typically there will be no liquid options prices struck on $\widetilde{S}_f$, and we will need to construct the implied volatility for these ourselves. Denote by $\Phi^{IMP}(\widetilde{S}_f(t),t)$ the implied volatility for calls struck on $\widetilde{S}_f$, which is used subsequently to obtain the local volatility functional in the diffusion equations. There are many techniques in the literature for solving the ill-posed function approximation $\Phi^{IMP}(\widetilde{S}_f(t),t) \to \Phi(\widetilde{S}_f(t),t)$, once $\Phi^{IMP}(\widetilde{S}_f(t),t)$ has been found. These include a PDE approach (ABR[97]) and spline functional methods (CLV[99]).

We modify an approach pioneered by ABR[97] to find the local volatility using PDE methods, which is obtained using well-known results[10] for the underlying processes S_f and X. ABR[97] show that, provided observable market prices for call and put options for all strikes and maturities exist or can be interpolated/extrapolated, and provided the underlying process (foreign stock, FX rate) follows geometric Brownian motion, the instantaneous local volatility functional satisfies the following PDE, written in terms of the market implied volatility smile $\theta(S_t, t; K, T)$.

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[8] A convertible that is not callable for a certain number of months or years is said to have hard call protection. This is in contrast to soft call protection, when the convertible becomes callable after the underlying equity (here $\widetilde{S}_f$) trades at a predetermined level above initial price.

[9] This is typically true for equity and FX up to 5 years. FX implied volatilities from front office systems such as MUREX are available typically up to 2 years, and longer maturities up to 5 years can be obtained from brokers. Beyond 5 years, however, FX options prices are hard to come by, and the methodology suggested here is less useful.

[10] Namely, that the probability density function for the underlying process at some intermediate time satisfies the Fokker-Planck forward equation (see ABR[97], pp. 9, equation 7).

$$\sigma^2(K,T) = \frac{2\frac{\partial\theta}{\partial T} + \frac{\theta}{T-t} + 2K\{r(T) - \gamma(T)]\frac{\partial\theta}{\partial K}}{K^2\left[\frac{\partial^2\theta}{\partial K^2} - d_+\sqrt{T-t}\left(\frac{\partial\theta}{\partial K}\right)^2 + \frac{1}{\theta}\left(\frac{1}{K\sqrt{T-t}} + d_+\frac{\partial\theta}{\partial K}\right)^2\right]}$$

ABR[97] use a one-factor Crank-Nicholson technique to solve for the local volatility by first stating the discretization to the pricing PDE for any contingent claim. They then calibrate to bond prices and asset forwards assuming the existence of a continuous term structure of interest rates and dividend yields. They then solve for the Arrow-Debreu prices implied by call option prices and use these to back out the local volatility functional via transforming the original discretization scheme. Their method reports very high accuracy for calibration to the vanilla options smile. Our modification to the ABR[97] approach consists of a correction to the basic discretization scheme for the one-factor Crank-Nicholson method[11] which omits a term on the left hand side of the pricing equation. This insubstantial error affects many of the equations in the ABR[97], which have to be adjusted, although the impact on the pricing is most probably quite small.

To obtain $\Phi^{IMP}(\widetilde{S}_f(t),t)$ for the synthetic call option struck on $\widetilde{S}_f$, recall that the price can be expressed as an expectation over the risk-neutral domestic measure:

$$C(\widetilde{S}_f, T, \Phi^{IMP}) = e^{-r_d T} E\left[(S_f(T)X(T) - K, 0)^+ \mid \Im(0)\right] \qquad (6)$$

$$= e^{-r_d T}\int_{\frac{K}{K'}}^{\infty}\left(\int_{K'}^{\infty}(S_f(T)X(T) - K)\,\psi(S_f(T) \mid X(T); X(0), S_f(0))dS_f\right)\psi(X(T) \mid X(0))dX$$

where $\psi(S_f(T) \mid X(T); X(0), S_f(0))$ is the conditional *lognormal* probability density function for S_f given X at maturity, given by:

$$\psi(S_f(T) \mid X(T); X(0), S_f(0)) = \frac{\psi(S_f(T)X(T) \mid X(0), S_f(0))}{\psi(X(T) \mid X(0))} = \frac{\psi(\widetilde{S}_f(T) \mid \widetilde{S}_f(0))}{\psi(X(T) \mid X(0))}$$

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[11] We refer to equation (21) in ABR[97], which fails to put a $-\theta\hat{r}_j\Delta t$ term multiplying $H_{i,j+1}$ on the right-hand side of the equation. The $H_{i,j}$ term on the left-hand side is also incorrectly multiplied, by $\hat{r}_j\Delta t$ instead of $(1-\theta)\hat{r}_j\Delta t$.

and $\psi(S_f(T)X(T)|\,X(0),S_f(0))$ is the joint pdf for S_f and X, and can be obtained from the lognormal diffusion representation for $\tilde{S}_f$:

$$\psi(S_f(T)X(T)|\,X(0),S_f(0)) = \frac{1}{S_f X(\sigma_X^{IMP} + \sigma_f^{IMP})\sqrt{2T\pi}} \exp\left[\frac{-[\ln(\frac{S_f(T)X(T)}{S_f(0)X(0)}) - ((r_d - \gamma) - \frac{1}{2}(\sigma_X^{IMP} + \sigma_f^{IMP})^2)T]^2}{2T\left(\left\|\sigma_X^{IMP}\right\|^2 + 2\rho_{X,f}\left\|\sigma_X^{IMP}\right\|\left\|\sigma_f^{IMP}\right\| + \left\|\sigma_f^{IMP}\right\|^2\right)}\right] \tag{7}$$

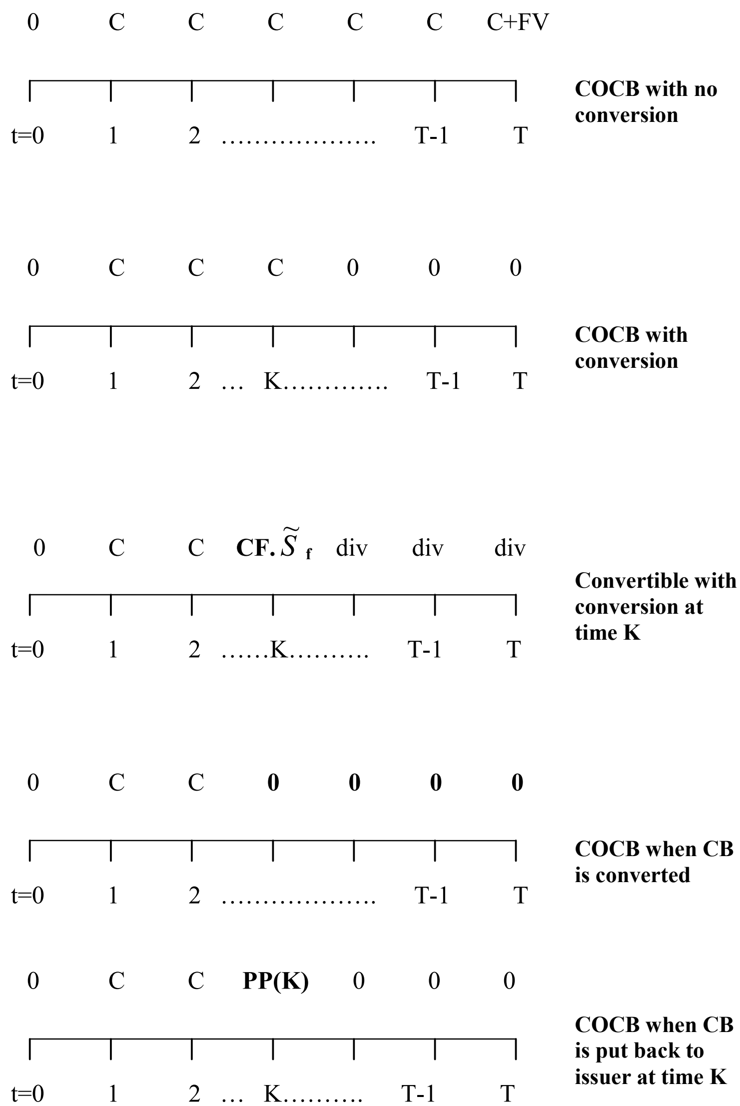
with:

$$\psi(X(T)\,|\,X(0)) = \frac{1}{X\sigma_X^{IMP}\sqrt{2\pi T}} \exp\left[\frac{-\left[\ln(X(T)/X(0)) - \left(r_d - r_f - \frac{1}{2}\left\|\sigma_X^{IMP}\right\|^2\right)\sqrt{T}\right]^2}{2\left\|\sigma_X^{IMP}\right\|^2}\right] \tag{8}$$

As the double integral depends on the smile for foreign equity in foreign currency and on the exchange rate, the numerical solution to the integral gives the implied-implied volatility for the asset $\tilde{S}_f$.

3.5 Credit Risk

Credit risk is included by introducing the cash only part of the convertible bond (COCB), a synthetic security discussed first in TF[98]. The payoff of the COCB depends on the underlying convertible security, and the holder of the COCB is paid all the fixed income flows provided the convertible remains unconverted, and any put cash flows if the holder of the convertible decides to put the bond back to the issuer.



As no call notice periods and instantaneous conversion is assumed, once the convertible price reaches the call price plus accrued interest, it is imposed that at call the COCB is killed as the convertible is converted into foreign equity. As the COCB fixed income cash flows are tied in with the issuer's ability to procure the necessary funds to honour its coupon/face repayment and put payment, the appropriate discount rate for the COCB in the pricing equation is the risk free rate plus a credit spread that the market attributes to the issuer's nonconvertible corporate bonds.

Following the notation of TF[98], we use *v* to denote the COCB and *u* to denote the convertible bond. The COCB for the international convertible bond then follows the pricing equation

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial r}(\theta-ar)+\frac{\partial v}{\partial \tilde{S}_f}(r-\gamma)\tilde{S}_f+\frac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 v}{\partial \tilde{S}_f^2}+\frac{1}{2}\frac{\partial^2 v}{\partial r^2}w^2 r$$
$$+\frac{\partial^2 v}{\partial \tilde{S}_f\partial r}\tilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\tilde{S},r}-(r+r_c)v=0 \qquad (9)$$
$$\textit{with}\ \text{LHS}\le 0\ \text{ for }\tilde{S}_f(t)>\tilde{S}_{free-boundary}$$

where r_c is some spread above treasury rates that is *related,* but not necessarily the same as, the spread on nonconvertible corporate debt by the same issuer. TF[98] use the nonconvertible debt spread to discount the COCB. This approach is prone to error because corporate debt is nearly always senior to convertible debt. Thus the add-on (to the risk-free rate) that applied to the COCB, which we'll call "pure spread", for want of a better word, can well be above the senior corporate bond's credit spread.

The residual security $u - v$ is the equity component of the convertible bond, i.e. that part of the convertible whose payoff is contingent on the original convertible being converted. Following TF[98], this should be discounted at the risk-free rate as the issuer of the international convertible bond can always deliver its own equity even when it is worthless. As $u - v$ also depends on $\tilde{S}_f$ and *r*, the pricing PDE that $u - v$ satisfies is

$$\frac{\partial(u-v)}{\partial t}+\frac{\partial(u-v)}{\partial r}(\theta-ar)+\frac{\partial(u-v)}{\partial \tilde{S}_f}(r-\gamma)\tilde{S}_f+\tfrac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2(u-v)}{\partial \tilde{S}_f^2}$$
$$+\tfrac{1}{2}\frac{\partial^2(u-v)}{\partial r^2}w^2 r+\frac{\partial^2(u-v)}{\partial \tilde{S}_f\partial r}\tilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\tilde{S},r}-r(u-v)=0 \qquad (10)$$
$$\text{with LHS}\le 0\ \text{ for }\tilde{S}_f(t)>\tilde{S}_{free-boundary}$$

Rearranging the above leads to:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}(\theta - ar)+\frac{\partial u}{\partial \tilde{S}_f}(r-\gamma)\tilde{S}_f+\tfrac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 u}{\partial \tilde{S}_f^2}+\tfrac{1}{2}\frac{\partial^2 u}{\partial r^2}w^2 r$$
$$+\frac{\partial^2 u}{\partial \tilde{S}_f \partial r}\tilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\tilde{S},r}-ru=-rv+\frac{\partial v}{\partial t}+\frac{\partial v}{\partial r}(\theta - ar)+\frac{\partial v}{\partial \tilde{S}_f}(r-\gamma)\tilde{S}_f$$
$$+\tfrac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 v}{\partial \tilde{S}_f^2}+\tfrac{1}{2}\frac{\partial^2 v}{\partial r^2}w^2 r+\frac{\partial^2 v}{\partial \tilde{S}_f \partial r}\tilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\tilde{S},r}$$

The left hand side of the equation above is equal to $r_c v$. Therefore, the international convertible bond pricing PDE can be rearranged to yield

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}(\theta - ar(t))+\frac{\partial u}{\partial \tilde{S}_f}(r-\gamma)\tilde{S}_f+\tfrac{1}{2}(\sigma_X^2+2\rho_{f,X}\sigma_X\sigma_f+\sigma_f^2)\tilde{S}_f^2\frac{\partial^2 u}{\partial \tilde{S}_f^2}+\tfrac{1}{2}\frac{\partial^2 u}{\partial r^2}w^2 r$$
$$+\frac{\partial^2 u}{\partial \tilde{S}_f \partial r}\tilde{S}_f(\sigma_X+\sigma_f)w\sqrt{r}\rho_{\tilde{S},r}-ru-r_c v=0 \qquad (11)$$

3.6 Boundary and Initial conditions

3.6.1 COCB case

The coupling in the equations (11) is apparent through the term $r_c v$, which must be available at the time of solving for the new time step u^n given the value at u^{n+1}. Crank-Nicholson with PSOR is used to solve for u^n above, but using the previous value for v, i.e. v^{n+1}. The algorithm for v is a separate finite difference with v^n obtained from v^{n+1} using the Crank-Nicholson scheme with the boundary/initial conditions obtained from the solution for the convertible bond, u. That is, the boundary/initial conditions for v are given by:

Initial conditions:

$$v(\tilde{S}_f(T), r(T), T)=\begin{cases} 0 \text{ if the bond was converted prior to or at maturity} \\ \max(PP(T)+AcInt(T), \text{Face Value}+AcInt(T)) \\ =\max(PP(T)+\text{coupon}, FV+\text{coupon}) \end{cases} \qquad (12)$$

i.e. $v(\tilde{S}_f(T), r(T), T)=(\max[PP, FV]+\text{Coupon})I_{n\tilde{S}_f<\max(PP,FV)}$

Boundary conditions:

$v^n_{m,l}(\tilde{S}_{\max})=0$ where n refers to the time step, m refers to the asset step, and l to the interest rate step. The (coupled) boundary conditions that are used on the grid are:

$$v^n_{m,l} = \begin{cases} PP(t) + AcInt(t) \text{ if } u^n_{m,l} \le PP(t) + AcInt(t) \\ 0 \text{ if } u^n_{m,l} \ge CP(t) + AcInt(t) \\ 0 \text{ if } u^n_{m,l} \le nS + AcInt(t) = n(m\Delta s) + AcInt(t) \end{cases} \tag{13}$$

$v^n_{m,l} \le u^n_{m,l} \quad \forall\, m,l$ (the contemporaneous value of u is an upper bound for v).

These are combined with the convertible bond initial/boundary conditions for *u* below.

3.6.2 CB case

Initial condition:

$$u(\tilde{S}_f(T), r(T), T) = \max(FV + Coupon(T), PP(T) + Coupon(T), \min(CR\tilde{S}_f(T), CP(T) + Coupon)$$

Boundary condition:

$$\tilde{S}_f = \tilde{S}_{max} \qquad u(\tilde{S}_{max}, r, t) = \begin{cases} CR.\tilde{S}_{max} & \text{for u non - callable} \\ 0 & \text{for u callable} \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial s}\bigg|_{\tilde{S}_{max}} = \begin{cases} n & \text{for u non - callable} \\ 0 & \text{for u callable} \end{cases}$$

$$\frac{\partial^2 u}{\partial S^2} = 0 \text{ everywhere on } \tilde{S} = \tilde{S}_{max}$$

We also impose the continuous upper and lower constraints:

$$u^n_{m,l} = max[PP(t) + AcInt(t), Min[CP(t) + Acint(t), CR.\tilde{S}_f + AcInt(t)]] \tag{14}$$

The solution proceeds as follows: first u^{T-1} is solved for using CN with u^T, and v^T, known from initial conditions. Then solve for v^{T-1} using the boundary conditions for v^{T-1}, which depends on u^{T-1}, from above. In this way, iterate between *u*, and *v*, and solve until u^0 is obtained.

4. Numerical Methods

For parabolic PDE's, the time and space dynamics do not evolve jointly. That is, there are no mixed time and so-called "space" (equity and interest rate) partial derivatives in the PDE. We can therefore solve the convertible bond problem in two distinct steps: first we construct a grid corresponding to the space discretization; then we proceed to the second step via a backward time-stepping procedure where early call, put, discrete dividends, and any other time-dependent jump conditions are applied in the successive over-relaxation routine. This starts with the last time-step value of the convertible bond as an initial iterate, and continues the iteration towards a new solution, i.e. the previous time step value.

Formally, we have the relationship

$$\frac{\partial u}{\partial t} = -Du + ru - r_c v = D_U u - r_c v \text{ with } D_U = -D + r$$

where:

$$D = \frac{\partial}{\partial r}(\theta - ar(t)) + \frac{\partial}{\partial \widetilde{S}_f}(r-\gamma)\widetilde{S}_f + \frac{1}{2}(\sigma_X^2 + 2\rho_{f,X}\sigma_X\sigma_f + \sigma_f^2)\widetilde{S}_f^2 \frac{\partial^2}{\partial \widetilde{S}_f^2}$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial r^2} w^2 r + \frac{\partial^2}{\partial \widetilde{S}_f \partial r}\widetilde{S}_f(\sigma_X + \sigma_f) w \sqrt{r}\rho_{\widetilde{S},r}$$

The time, equity, and interest rate grid points are indexed respectively as *m, n,* and *l*, with corresponding step sizes $\Delta t = T/N$, $\Delta r = r_{max}/L$, $\Delta\widetilde{S}_f = \widetilde{S}_f^{max}/M$, for *N, M,* and *L* taking values in the set of positive integers. The notation $u_{m,l}^n$ then corresponds to the value of the convertible at the grid points $u(n.\Delta t, m.\Delta\widetilde{S}_f, l.\Delta r)$. In the following it is emphasised that the time index *n* refers to calendar time, not inverse time[12].

The Crank-Nicholson two-step procedure is then cast as a set of two ordinary differential equations:

$$\frac{d\mathbf{u}(t)}{dt} = \tfrac{1}{2}\hat{D}_U u(t) + \tfrac{1}{2}\hat{D}_U u(t+\Delta t) + r_c \mathrm{v} \tag{15}$$

$$\text{where } \frac{du_{m,l}}{dt} = \frac{u_{m,l}^{t+\Delta t} - u_{m,l}^{t}}{\Delta t} = \tfrac{1}{2}\sum_{i,j}\hat{D}_{U,m,l,i,j}u_{i,j}(t) + \tfrac{1}{2}\sum_{i,j}\hat{D}_{U,m,l,i,j}u_{i,j}(t+\Delta t) + r_c v_{m,l}(t)$$

Here $\hat{D}_{U;m,l,i,j}$ refers to the space discretization matrix representing (mostly second order accurate) *r* and *S* discretization of the partial differentials. The *i* and *j* indices refer to the lag and lead terms in the space discretization, and are typically in the range $m \pm 2, \text{and } l \pm 2$, depending on whether centred (in the centre of the grid) or non-centred (in the boundaries) schemes are employed.

The relevant matrix form of the problem is:

$$\left(\frac{1}{\Delta t}\mathbf{1}+\frac{1}{2}\hat{D}_U\right)\mathbf{v}^{\mathbf{n}} = \left(\frac{1}{\Delta t}\mathbf{1}-\frac{1}{2}\hat{D}_U\right)\mathbf{u}^{\mathbf{n+1}} + r_c\mathbf{v}^{\mathbf{n+1}} \qquad (16)$$

As the final payoff for the convertible and the COCB is known, the problem consists in solving the equation above backward in time to obtain today's values, subject to boundary conditions for the coupled PDE's discussed before. Truncation points for the grid are also chosen with the intent to delimit the grid to those $(\widetilde{S}_f, r)$ regions that are of interest. Sensible choices are $\widetilde{S}_f^{\max}$ at 3 standard deviations from its expected future value (typically in the magnitude of 1200% of initial value for a five year convertible bond), $\widetilde{S}_f^{\min} = 0$, $r^{\max}$=20-25%, and $r^{\min}$=0.

The solution to the matrix equations and the underlying PDE depends very much on the choice of the space discretization matrix $\hat{D}$ above. High order accuracy for the finite difference approximations depends on the choice of scheme at the boundaries and the interior of the grid. In the appendix we consider the discretized PDE in various regions of the grid and report on the accuracy of the schemes. Note that it is straightforward to impose that the credit spread is some function $r_c\left(\widetilde{S}_f(n.\Delta t, m.\Delta\widetilde{S}_f, l.\Delta r)\right)$, dependent on the location in the grid. The effect of negative correlation between credit spreads and equity can then be incorporated.

[12] By inverse time we refer to time evolving backwards away from the maturity of the convertible.

5. Conclusion

This paper has presented a comprehensive theoretical framework for pricing convertible bonds in the presence of interest rate, equity, credit, and FX risk in a two factor model incorporating most of the contractual features. We have shown that FX risk for international convertible bonds does not introduce an extra factor in the PDE used to price it. Furthermore, we propose a procedure to back out the price of vanilla options and hence the local volatility for the foreign equity in domestic currency. The model we propose accounts for convertible credit risk by extending the TF[98] model for interest rate risk. Suggestions are made to make the treatment of credit risk more robust, in particular to capture the negative correlation between equity and credit spreads, by imposing a functional dependence of the spread in the various regions (i.e. boundaries) on the grid. To this end, and to provide discretization accuracy results, we present the complete discretization scheme that characterises the grid in all regions.

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Appendix 1 on the Discretization Schemes

(i) For $m = 0$, $l = 0$, which corresponds to $S = 0$, $r = 0$, the partial differential equation becomes

$$\frac{\partial u}{\partial t} = -\theta \frac{\partial u}{\partial r} + r_c v$$

The spatial discretization is then given by

$$\frac{\mathrm{du}_{0,0}}{\mathrm{dt}} = \frac{\mathrm{u}_{0,0}^{\mathrm{n+1}} - \mathrm{u}_{0,0}^{\mathrm{n}}}{\tau_{\mathrm{n}}} = -\frac{\theta}{2} . \frac{-3\mathrm{u}_{0,0}^{\mathrm{n+1}} + 4\mathrm{u}_{0,1}^{\mathrm{n+1}} - \mathrm{u}_{0,2}^{\mathrm{n+1}}}{2\Delta_{\mathrm{r}}} - \frac{\theta}{2} . \frac{-3\mathrm{u}_{0,0}^{\mathrm{n}} + 4\mathrm{u}_{0,1}^{\mathrm{n}} - \mathrm{u}_{0,2}^{\mathrm{n}}}{2\Delta_{\mathrm{r}}} + \mathrm{r_c}\mathrm{v}_{0,0}^{\mathrm{n+1}}$$

This scheme possesses second order accuracy in both Δ_S and Δ_r (see TR[00])

(ii) For $m = 0$, $l = 1, \ldots, L-1$, which corresponds to $S = 0$ and $r \neq 0$, the partial differential equation reduces to:

$$\frac{\partial u}{\partial t} = -\frac{1}{2} w^2 r \frac{\partial^2 u}{\partial r^2} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v$$

The spatial discretization is then given by

$$\frac{du_{0,l}^{n}}{dt} = \frac{\mathrm{u}_{0,l}^{\mathrm{n+1}} - \mathrm{u}_{0,l}^{\mathrm{n}}}{\tau_{\mathrm{n}}} = -\frac{1}{2} w^2 r_l \left(\frac{u_{0,l+1}^{n+1} - 2u_{0,l}^{n+1} + u_{0,l-1}^{n+1}}{2\Delta r^2} \right) - \frac{1}{2} w^2 r_l \left(\frac{u_{0,l+1}^{n} - 2u_{0,l}^{n} + u_{0,l-1}^{n}}{2\Delta r^2} \right)$$
$$+ (\frac{ar_l - \theta}{2}) \left(\frac{u_{0,l+1}^{n+1} - u_{0,l-1}^{n+1}}{2\Delta r} \right) + (\frac{ar_l - \theta}{2}) \left(\frac{u_{0,l+1}^{n} - u_{0,l-1}^{n}}{2\Delta r} \right) + r_l \left(\frac{u_{0,l}^{n+1} + u_{0,l}^{n}}{2} \right) + r_c v_{0,l}^{n+1}$$

This is second order accurate in both ΔS and Δr.

(iii) For $m = 0$, $l = L$ ($S = 0$, $r = rmax \neq 0$), the partial differential equation simplifies to:

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2} w^2 r \frac{\partial^2 u}{\partial r^2} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v$$

The spatial discretization is then given by

$$\frac{du_{0,L}}{dt} = \frac{\mathrm{u}_{0,\mathrm{L}}^{\mathrm{n+1}} - \mathrm{u}_{0,\mathrm{L}}^{\mathrm{n}}}{\tau_{\mathrm{n}}} = -\tfrac{1}{2} w^2 r_l \tfrac{1}{2} \left(\frac{u_{0,L}^{n+1} - 2u_{0,L-1}^{n+1} + u_{0,L-2}^{n+1}}{\Delta_r^2} \right) - \tfrac{1}{2} w^2 r_l \tfrac{1}{2} \left(\frac{u_{0,L}^{n} - 2u_{0,L-1}^{n} + u_{0,L-2}^{n}}{\Delta_r^2} \right)$$
$$+ (ar_l - \theta)\tfrac{1}{2} \left(\frac{3u_{0,L}^{n+1} - 4u_{0,L-1}^{n+1} + u_{0,L-2}^{n+1}}{2\Delta_r} \right) + (ar_l - \theta)\tfrac{1}{2} \left(\frac{3u_{0,L}^{n} - 4u_{0,L-1}^{n} + u_{0,L-2}^{n}}{2\Delta_r} \right)$$
$$+ r_l \tfrac{1}{2} (u_{0,L}^{n+1} + u_{0,L}^{n+1}) + r_c v_{0,L}$$

which is second order accurate in ΔS and Δr.

(iv) For $m = 1, \ldots, M$-1, $l = 0$ ($S \neq 0$, $r = 0$), the partial differential equation can be written as

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2}(\sigma_X^2 + 2\rho_{f,X}\sigma_X\sigma_f + \sigma_f^2)\widetilde{S}_f^2 \frac{\partial^2 u}{\partial \widetilde{S}_f^2} + \gamma S \frac{\partial u}{\partial \widetilde{S}_f} - \theta \frac{\partial u}{\partial r} - r_c v$$

$$= -\tfrac{1}{2}\widetilde{\sigma}_f^2(\widetilde{S}_f, t)\widetilde{S}_f^2 \frac{\partial^2 u}{\partial \widetilde{S}_f^2} + \gamma S \frac{\partial u}{\partial \widetilde{S}_f} - \theta \frac{\partial u}{\partial r} - r_c v$$

where $\widetilde{\sigma}_f^2(\widetilde{S}_f, t)$ refers to the local volatility for foreign equity in domestic currency, obtained numerically from the integration *(add ref to section)* . The spatial discretization is then given by

$$\frac{du_{m,0}}{dt} = \frac{\mathrm{u}_{0,1}^{\mathrm{n}+1} - \mathrm{u}_{0,1}^{\mathrm{n}}}{\tau_{\mathrm{n}}} = -\tfrac{1}{2}\widetilde{\sigma}_f^2 \widetilde{S}_m^2 \tfrac{1}{2}\left(\frac{u_{m+1,0}^{n+1} - 2u_{m,0}^{n+1} + u_{m-1,0}^{n+1}}{\Delta_S^2}\right) - \tfrac{1}{2}\widetilde{\sigma}_f^2 \widetilde{S}_m^2 \tfrac{1}{2}\left(\frac{u_{m+1,0}^{n} - 2u_{m,0}^{n} + u_{m-1,0}^{n}}{\Delta_S^2}\right)$$

$$+ \gamma\widetilde{S}_m \tfrac{1}{2}\left(\frac{u_{m+1,0}^{n+1} - u_{m-1,0}^{n+1}}{2\Delta_S}\right) + \gamma\widetilde{S}_m \tfrac{1}{2}\left(\frac{u_{m+1,0}^{n} - u_{m-1,0}^{n}}{2\Delta_S}\right)$$

$$- \theta \tfrac{1}{2}\left(\frac{-3u_{m,0}^{n+1} + 4u_{m,1}^{n+1} - u_{m,2}^{n+1}}{2\Delta_r}\right) - \theta \tfrac{1}{2}\left(\frac{-3u_{m,0}^{n} + 4u_{m,1}^{n} - u_{m,2}^{n}}{2\Delta_r}\right) - r_c v_{m,0}$$

which is second order accurate in ΔS and Δr.

(v) For $m = 1, \ldots, M$-1, $l = 1, \ldots, L$-*1* ($S \neq 0$, $r \neq 0$) the partial differential equation can be written as:

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2}\widetilde{\sigma}_f^2(\widetilde{S}_m, t)\widetilde{S}_m^2 \frac{\partial^2 u}{\partial \widetilde{S}^2} - \widetilde{S}_m \widetilde{\sigma}_f(\widetilde{S}_m, t) w\sqrt{r}\rho_{\widetilde{S},r} \frac{\partial^2 u}{\partial r \partial \widetilde{S}} - \tfrac{1}{2}w^2 r \frac{\partial^2 u}{\partial r^2}$$

$$+ (\gamma - r)S\frac{\partial u}{\partial \widetilde{S}} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v$$

The spatial discretization is then given by:

$$\frac{du_{m,l}}{dt} = \frac{\mathrm{u}^{\mathrm{n+1}}_{\mathrm{m,l}} - \mathrm{u}^{\mathrm{n}}_{\mathrm{m,l}}}{\tau_{\mathrm{n}}} = -\tfrac{1}{2}\tilde{\sigma}^2_f(\tilde{S}_m,t)\tilde{S}^2_m \tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,l} - 2u^{n+1}_{m,l} + u^{n+1}_{m-1,l}}{\Delta^2_S}\right)$$

$$-\tfrac{1}{2}\tilde{\sigma}^2_f(\tilde{S}_m,t)\tilde{S}^2_m \tfrac{1}{2}\left(\frac{u^{n}_{m+1,l} - 2u^{n}_{m,l} + u^{n}_{m-1,l}}{\Delta^2_S}\right)$$

$$-\tilde{S}_m\tilde{\sigma}_f(\tilde{S}_m,t)w\sqrt{r_l}\,\rho_{\tilde{S},r}\ \tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,l+1} - u^{n+1}_{m+1,l-1} - u^{n+1}_{m-1,l+1} + u^{n+1}_{m-1,l-1}}{4\Delta_r\Delta_S}\right)$$

$$-\tilde{S}_m\tilde{\sigma}_f(\tilde{S}_m,t)w\sqrt{r_l}\,\rho_{\tilde{S},r}\ \tfrac{1}{2}\left(\frac{u^{n}_{m+1,l+1} - u^{n}_{m+1,l-1} - u^{n}_{m-1,l+1} + u^{n}_{m-1,l-1}}{4\Delta_r\Delta_S}\right)$$

$$-\tfrac{1}{2}w^2r_l\ \tfrac{1}{2}\left(\frac{u^{n+1}_{m,l+1} - 2u^{n+1}_{m,l} + u^{n+1}_{m,l-1}}{\Delta^2_r}\right) - \tfrac{1}{2}w^2r_l\ \tfrac{1}{2}\left(\frac{u^{n}_{m,l+1} - 2u^{n}_{m,l} + u^{n}_{m,l-1}}{\Delta^2_r}\right)$$

$$+(\gamma - r_l)S_m\ \tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,l} - u^{n+1}_{m-1,l}}{2\Delta_S}\right) + (\gamma - r_l)S_m\ \tfrac{1}{2}\left(\frac{u^{n}_{m+1,l} - u^{n}_{m-1,l}}{2\Delta_S}\right)$$

$$+(ar_l - \theta)\tfrac{1}{2}\left(\frac{u^{n+1}_{m,l+1} - u^{n+1}_{m,l-1}}{2\Delta_r}\right) + (ar_l - \theta)\tfrac{1}{2}\left(\frac{u^{n}_{m,l+1} - u^{n}_{m,l-1}}{2\Delta_r}\right) + r_l\ \tfrac{1}{2}(u^{n+1}_{m,l} + u^{n}_{m,l}) + r_c v_{m,l}$$

which is second order accurate in ΔS and Δr.

(vi) For $m = 1, \ldots, M$-1, $l = L$ ($S \neq 0$, $r = rmax \neq 0$) the partial differential equation can be written as:

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2}\tilde{\sigma}^2_f(\tilde{S}_m,t)\tilde{S}^2_m\frac{\partial^2 u}{\partial\tilde{S}^2} - \tilde{S}_m\tilde{\sigma}_f(\tilde{S}_m,t)w\sqrt{r}\rho_{\tilde{S},r}\frac{\partial^2 u}{\partial r\partial\tilde{S}} - \tfrac{1}{2}w^2r\frac{\partial^2 u}{\partial r^2} + (\gamma - r)S\frac{\partial u}{\partial\tilde{S}} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v$$

The spatial discretization is then given by:

$$\frac{du_{m,L}}{dt} = \frac{\mathrm{u}^{\mathrm{n+1}}_{\mathrm{m,L}} - \mathrm{u}^{\mathrm{n}}_{\mathrm{m,L}}}{\tau_{\mathrm{n}}} = -\tfrac{1}{2}\tilde{\sigma}_f^2(\tilde{S}_m, t)\tilde{S}_m^2 \tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,L} - 2u^{n+1}_{m,L} + u^{n+1}_{m-1,L}}{\Delta_S^2}\right)$$
$$-\tfrac{1}{2}\tilde{\sigma}_f^2(\tilde{S}_m, t)\tilde{S}_m^2 \tfrac{1}{2}\left(\frac{u^{n}_{m+1,L} - 2u^{n}_{m,L} + u^{n}_{m-1,L}}{\Delta_S^2}\right)$$
$$-\tilde{S}_m\tilde{\sigma}_f(\tilde{S}_m, t)w\sqrt{r_L}\rho_{\tilde{S},r}\tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,L} - u^{n+1}_{m+1,L-1} - u^{n+1}_{m-1,L} + u^{n+1}_{m-1,L-1}}{2\Delta_r\Delta_S}\right)$$
$$-\tilde{S}_m\tilde{\sigma}_f(\tilde{S}_m, t)w\sqrt{r_L}\rho_{\tilde{S},r}\tfrac{1}{2}\left(\frac{u^{n}_{m+1,L} - u^{n}_{m+1,L-1} - u^{n}_{m-1,L} + u^{n}_{m-1,L-1}}{2\Delta_r\Delta_S}\right)$$
$$-\tfrac{1}{2}w^2 r_L \tfrac{1}{2}\left(\frac{u^{n+1}_{m,L} - 2u^{n+1}_{m,L-1} + u^{n+1}_{m,L-2}}{\Delta_r^2}\right) - \tfrac{1}{2}w^2 r_L \tfrac{1}{2}\left(\frac{u^{n}_{m,L} - 2u^{n}_{m,L-1} + u^{n}_{m,L-2}}{\Delta_r^2}\right)$$
$$+(\gamma - r_L)S_m \tfrac{1}{2}\left(\frac{u^{n+1}_{m+1,L} - u^{n+1}_{m-1,L}}{2\Delta_S}\right) + (\gamma - r_L)S_m \tfrac{1}{2}\left(\frac{u^{n}_{m+1,L} - u^{n}_{m-1,L}}{2\Delta_S}\right)$$
$$+(ar_L - \theta)\tfrac{1}{2}\left(\frac{3u^{n+1}_{m,L} - 4u^{n+1}_{m,L-1} + u^{n+1}_{m,L-2}}{2\Delta_r}\right) + (ar_L - \theta)\tfrac{1}{2}\left(\frac{3u^{n}_{m,L} - 4u^{n}_{m,L-1} + u^{n}_{m,L-2}}{2\Delta_r}\right)$$
$$+\tfrac{1}{2}r_L(u^{n+1}_{m,L} + u^{n}_{m,L}) + r_c v_{m,L}$$

This is second order accurate in ΔS and first order accurate in Δr.

(vii) For $m = M$, $l = 0$, corresponding to $S = S_{max}$, $r = 0$ the partial differential equation can be written as:

$$\frac{\partial u}{\partial t} = \gamma S\frac{\partial u}{\partial S} - \theta\frac{\partial u}{\partial r} - r_c v \text{ as we assumed that at } \mathrm{S} = \mathrm{S}_{\max},\ \frac{\partial^2 u}{\partial \tilde{S}_f^2} = 0$$

The spatial discretization is then given by:

$$\frac{\mathrm{du}_{\mathrm{M,0}}}{\mathrm{dt}} = \frac{\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,0}} - \mathrm{u}^{\mathrm{n}}_{\mathrm{M,0}}}{\tau_{\mathrm{n}}} = \gamma\tilde{\mathrm{S}}_{\mathrm{M}}\tfrac{1}{2}\left(\frac{3\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,0}} - 4\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M-1,0}} + \mathrm{u}^{\mathrm{n+1}}_{\mathrm{M-2,0}}}{2\Delta\Delta}\right) + \gamma\tilde{\mathrm{S}}_{\mathrm{M}}\tfrac{1}{2}\left(\frac{3\mathrm{u}^{\mathrm{n}}_{\mathrm{M,0}} - 4\mathrm{u}^{\mathrm{n}}_{\mathrm{M-1,0}} + \mathrm{u}^{\mathrm{n}}_{\mathrm{M-2,0}}}{2\Delta\Delta}\right)$$
$$-\theta\tfrac{1}{2}\left(\frac{-3\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,0}} + 4\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,1}} - \mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,2}}}{2\Delta\Delta}\right) - \theta\tfrac{1}{2}\left(\frac{-3\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,0}} + 4\mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,1}} - \mathrm{u}^{\mathrm{n+1}}_{\mathrm{M,2}}}{2\Delta\Delta}\right) - \mathrm{r_c}\mathrm{v}_{\mathrm{M,0}}$$

which is second order accurate in ΔS and Δr

(viii) For $m = M$, $l = 1, \ldots, L-1$ ($S = Smax \neq 0$, $r \neq 0$) the partial differential equation simplifies to

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2} w^2 r \frac{\partial^2 u}{\partial r^2} + (\gamma - r)\tilde{S}_f \frac{\partial u}{\partial \tilde{S}_f} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v \text{ as } \frac{\partial^2 u}{\partial \tilde{S}_f \partial r} = 0, \text{ at the } \tilde{S}_f = \tilde{S}_{max} \text{ boundary}$$

The spatial discretization is then given by:

$$\frac{du_{M,l}}{dt} = -\tfrac{1}{2} w^2 r_l \tfrac{1}{2}\left(\frac{u^{n+1}_{M,l+1} - 2u^{n+1}_{M,l} + u^{n+1}_{M,l-1}}{\Delta_r^2}\right) - \tfrac{1}{2} w^2 r_l \tfrac{1}{2}\left(\frac{u^{n}_{M,l+1} - 2u^{n}_{M,l} + u^{n}_{M,l-1}}{\Delta_r^2}\right)$$
$$+ (\gamma - r_l)S_m \tfrac{1}{2}\left(\frac{3u^{n+1}_{M,l} - 4u^{n+1}_{M-1,l} + u^{n+1}_{M-2,l}}{2\Delta_S}\right) + (\gamma - r_l)S_m \tfrac{1}{2}\left(\frac{3u^{n}_{M,l} - 4u^{n}_{M-1,l} + u^{n}_{M-2,l}}{2\Delta_S}\right)$$
$$+ (ar_l - \theta)\tfrac{1}{2}\left(\frac{u^{n+1}_{M,l+1} - u^{n+1}_{M,l-1}}{2\Delta_r}\right) + (ar_l - \theta)\tfrac{1}{2}\left(\frac{u^{n}_{M,l+1} - u^{n}_{M,l-1}}{2\Delta_r}\right)$$
$$r_l \tfrac{1}{2}(u^{n+1}_{M,l} + u^{n}_{M,l}) + r_c v_{M,l}$$

This is second order accurate in ΔS and Δr

(ix) For $m = M$, $l = L$ ($S = Smax \neq 0$, $r = rmax \neq 0$) the partial differential equation becomes:

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2} w^2 r \frac{\partial^2 u}{\partial r^2} + (\gamma - r)\tilde{S}_f \frac{\partial u}{\partial S} + (ar - \theta)\frac{\partial u}{\partial r} + ru + r_c v$$

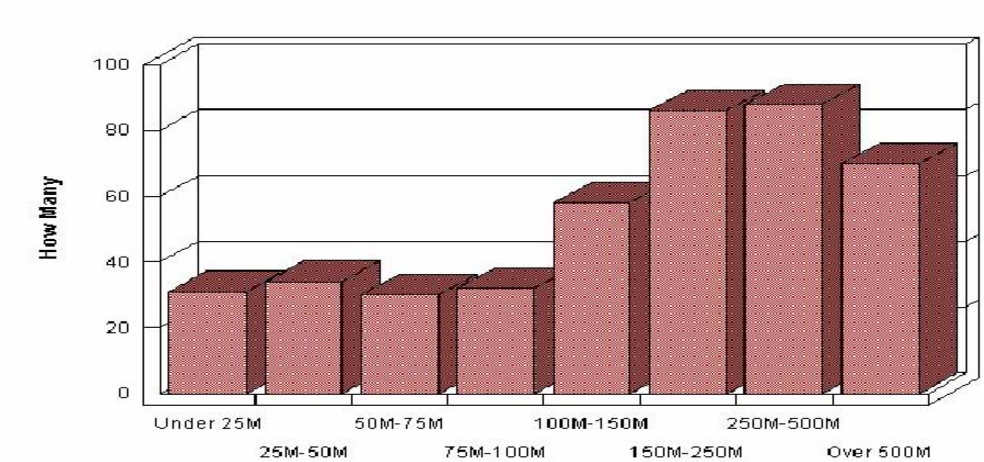
and the first order discretization is:

$$\frac{du_{M,L}}{dt} = -\tfrac{1}{2} w^2 r_L \tfrac{1}{2}\left(\frac{u^{n+1}_{M,L} - 2u^{n+1}_{M,L-1} + u^{n+1}_{M,L-2}}{\Delta_r^2}\right) - \tfrac{1}{2} w^2 r_L \tfrac{1}{2}\left(\frac{u^{n}_{M,L} - 2u^{n}_{M,L-1} + u^{n}_{M,L-2}}{\Delta_r^2}\right)$$
$$+ (\gamma - r_L)S_M \tfrac{1}{2}\left(\frac{3u^{n+1}_{M,L} - 4u^{n+1}_{M-1,L} + u^{n+1}_{M-2,L}}{2\Delta_S}\right) + (\gamma - r_L)S_M \tfrac{1}{2}\left(\frac{3u^{n}_{M,L} - 4u^{n}_{M-1,L} + u^{n}_{M-2,L}}{2\Delta_S}\right)$$
$$+ (ar_L - \theta)\tfrac{1}{2}\left(\frac{3u^{n+1}_{M,L} - 4u^{n+1}_{M,L-1} + u^{n+1}_{M,L-2}}{2\Delta_r}\right) + (ar_L - \theta)\tfrac{1}{2}\left(\frac{3u^{n}_{M,L} - 4u^{n}_{M,L-1} + u^{n}_{M,L-2}}{2\Delta_r}\right)$$
$$+ r_L \tfrac{1}{2}(u^{n+1}_{M,L} + u^{n}_{M,L}) + r_c v_{M,L}$$

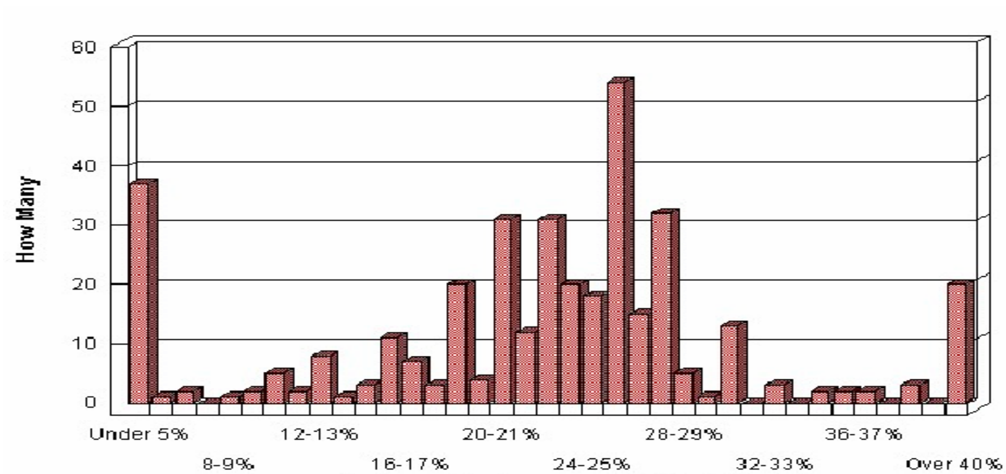
which is second order accurate in ΔS and first order accurate in Δr

Appendix 2:

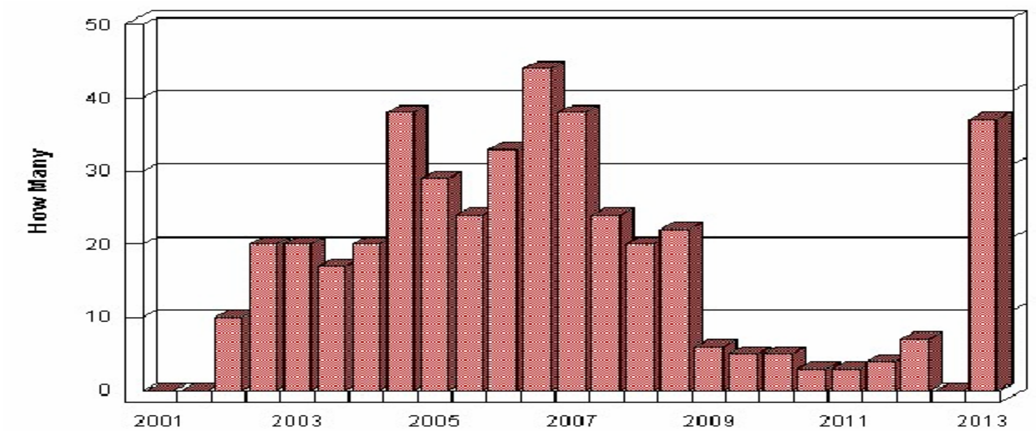
Convertible bonds by amount outstanding in the US:



Convertible bonds by initial premium in the US:



Convertible by maturity date in the US:



source:

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Appendix 3- term sheet for convertible with explicit FX sensitivity

Mitsubishi Bank resetable, mandatory, exchangeable, convertible security[13]

Fixed Issuance Terms

Issue Date:	Sep-95
Issue Size:	US$ 2 billion
Lead Manager:	Morgan Stanley
Coupon:	3%
Coupon Freq./dates:	Semi, Nov30/ May31
Maturity:	Nov. 2002
Conversion Period:	Apr.96- Nov.02

Early Mandatory Conversion

Beginning Date:	Nov. 30, 1998
Annually Onwards:	Nov. 30
Performance Criteria:	ADS> US$16.5

Reset Features:

1st Reset Date:	May 31, 1995
Annual Reset Date thereafter:	Nov. 30
Final Reset Date:	Nov. 30, 2002
Reset Calculation period:	20-day period excl. hols. Japan, NY, London t-15 days
Calculation type:	simple average
FX (Yen/US$) benchmark:	12pm NY Fed rate
Original Conversion Price:	US$22
Initial reset floor(%):	65%
Initial reset floor (US$):	US$13.61
Final reset floor (%):	50%
Final reset floor (US$):	US$10.47

[13] This term sheet can be found in C[98].