Analytically inducting option cash flows for Markovian interest rate models: A new application paradigm

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Abstract

This paper develops a new computational approach for general multi-factor Markovian interest rate models. The early exercise premium is derived for general American options. The option cash flows are decomposed into fast and slowly varying components. The fast components are option independent and derived analytically. The slow components are calculated by controlled expansion for finite time intervals. The option price is obtained by iterating the analytic expressions of one time interval. For one-factor models, the critical boundary for American options has a *universal* form near maturity. For American put stock options, analytic expressions are derived to approximate the critical boundary. The put price calculated from the boundary has relative precision better than 10^{-5} in all cases.

Modeling interest rate derivatives remains a challenge. Despite decades of effort, both model selection and practical computation are far from satisfactory. This paper studies the general class of multi-factor Markovian interest rate models and develops a new powerful approach to calculate interest rate derivatives efficiently and with controlled precision. This paper also presents analytic expressions to approximate the critical boundary of American put stock options and provides an effortless application to calculate the option price from the analytic expressions of the critical boundary.

In the early years, the interest rate models studied did not even match the prices of the underlying securities. Ho and Lee (Ho and Lee 1986) introduced the first so-called "arbitrage-free" interest rate model. Since then there have been various generalizations, such as the introduction of lognormal models and multiple random factors. However, these models are arbitrage free only for the underlying securities, e.g. US Treasuries. When interest rate derivatives are included, it is far more difficult to achieve arbitrage free status. First, there is a whole spectrum of interest rates of different maturities which form the yield curve. The different points on the yield curve, e.g. the short and long term interest rates, do no necessarily move in tandem. The principal mission of pricing financial derivatives as pioneered by Black and Scholes (Black and Scholes 1973) is to explore relations between different financial instruments. It is particularly desirable to price complicated instruments in terms of standard, often simple and liquid benchmark securities, provided that the underlying uncertainty has the same origin. However, if the underlying uncertain factors are different, they should all be included in the model. In the terminology of financial modeling, this is the requirement for completeness of the market model. Second, not only the volatility values but also the model selection should be determined by the market. The risk neutral valuation concept establishes that the options can be priced in the risk neutral world. However, there are considerable degrees of freedom left on the shape of the probability distribution even in the risk neutral world. These degrees of freedom have to be determined by the in and out of the money options. Furtheremore, there is no obvious reason why the market implied model should not change in time. Thus, multi-factor models are necessary for pricing short and long term interest rate derivatives. Furthermore, the market implied model should be chosen dynamically.

In this paper, we study the whole class of Markovian interest rate models for which the short rate is history independent. The Stochastic process for the short rate corresponding to HJM models (Heath, Jarrow and Morton 1992) can be either history dependent or independent. Thus, certain HJM models do not fall within the class of models considered by this paper.[1] However, it is not clear if there is any benefit from the extra complexity of the non-Markovian models. Indeed, there is no strong evidence to suggest that the price of US Treasuries is history dependent. For Markovian models, the short rate can be written as

$$r = r(t, x_1, x_2, \cdots), \tag{1}$$

where x_1, x_2 etc. are random variables. We do not impose any constraint on the functional form of $r(t, x_1, x_2, \cdots)$. Therefore, the results of this paper is not limited to any particular type of interest rate models, allowing dynamic model selection.

Since the life span of the underlying securities and their derivatives in interest rate models can be fairly long, multi-factor models present a numerical challenge for current computing power, and in the forseeable future. Furthermore, there is no simple way to estimate the numerical error incurred in existing calculations. The only way to estimate the numerical convergence in these approaches is to increase the numerical resolution and compare the results of different resolutions. This is very costly computationally and not very practical. The approach to be developed in this paper amounts to finding analytic expressions that can backward induct option cash flows for a finite time interval with controlled precision, as illustrated in Figure C. Two crucial results make this approach possible.

The first crucial result is the derivation of the early exercise premium (EEP) for American

[1]When the Stochastic process for the short rate is known, one can derive the Stochastic processes for the forward rates. Vice versa, the Stochastic process for the short rate can be derived from those of the forward rates.

options in general multi-factor Markovian interest rate models. With the help of EEP, the price of an American option is expressed as

$$P_a = P + \mathrm{EEP}, \tag{2}$$

where P and P_a are the prices of the American and European options with same maturity and same payoff at the maturity. The EEP representation has been derived for the American put option on a non-dividend paying stock, and for some simple term structure models (Jamshidian 1992, Chesney and Elliot and Gibson 1993). To the author's knowledge, it has not been derived for lognormal interest rate models. This is the first time that the EEP representation is derived for general interest rate models with the only constraint of being Markovian. The task of calculating EEP is mostly reduced to finding the critical boundary separating the American and European regions in the model parameter space. In the American region, including the critical boundary, the option should be exercised immediately. The critical boundary near the option maturity can be calculated analytically for general American options. In particular, the critical boundary for one-factor interest rate models has *universal* form near the maturity, e.g. equation (48). Away from the maturity, the critical boundary is calculated iteratively, along with the backward induction of option cash flows.

The second crucial result, which makes possible the analytical backward induction of option cash flows, is the decomposition of the European option cash flows into fast and slowly varying components. In this paper, a function $f(x)$ is said to be smooth if it is differentiable at least to the second order. And it is slowly varying if

$$\left|\delta x \frac{\partial f}{\partial x}\right| \ll |f(x)|, \quad \text{for } |\delta x| \ll 1. \tag{3}$$

Numerically, it is crucial to recognize that the option cash flows are not smooth and slowly varying functions, especially near the maturity. This is the reason why accurate option price is obtained only with large number of steps in any tree calculation (see Table II). This is also the reason why the straightforward finite difference methods (Hull 1997) do not work as well as expected. The solution developed in this paper is to separate out the fast varying components from the option cash flows. Fortunately, the fast varying components are only model dependent and option independent. In other words, they have the same expressions for all options within a model. Generally, they can be calculated analytically. The remaining smooth and slowly varying components contain all information about option specifics. They are calculated with controlled expansion from the appropriate differential equations. The approach of analytically backward inducting option cash flows will be demonstrated in great detail for general one-factor Markovian models. The results for multi-factor models are completely analogous. The early exercise premium is derived for multi-factor models in equation (45).

For American put option on a stock paying no dividend, we derive the asymptotic forms for the critical boundary both near and far away from the maturity. The crossover of the critical boundary between the two asymptotic limits is not available analytically. However, we have successfully derived approximate analytic expressions for the critical boundary, with relative precision better than 10^{-3} for the worst numerically fitted portion of the critical boundary. Calculating the put option price is reduced to an integration of time over the critical boundary. The results have relative precision better than 10^{-5}, i.e. the error is less than a fraction of a cent for option price of order 100. The accuracy and computational efficiency are illustrated in Table II. A C++ program is available from the author upon request.

The layout of the paper is as follows. In section I, the general multi-factor Markovian interest rate models are presented. To simplify notations, the presentation is limited to one and two factor models. In section II, the early exercise premium representation for general American options is derived. In section III, the decomposition of the option cash flows into fast and slowly varying components is presented. To conserve the notations, we limit our presentation to one-factor Markovian models. The fast components are derived analytically. The solution for the slowly varying components is included in Appendix B. In section IV, the results are applied to the Black-Scholes model of stock options. The boundary for the critical stock price and the pricing formula for American put options are derived. In section V, we summarize the practical steps for numerical implementation of interest rate models. The detailed derivation of the early exercise premium representation for the American options is presented in Appendix A. The expansion solution for the backward induction of the option cash flows is presented in Appendix B. The detailed results for American put stock option are included in Appendix C.

I. INTEREST RATE MODELS

We begin our discussion on general one-factor models, their different representation and relationship with other well known models. In the one-factor models, the interest rate is determined by a single random variable x,

$$r = r(t, x). \tag{4}$$

We choose r to be the short rate. As discussed in the Introduction, the short rate (4) will be assumed history independent. However, there is no constraint on the functional form of equation (4). Without losing generality, we can assume that the evolution of the random variable x in time is determined by the following Stochastic differential equation,

$$dx = -m(t, x)\, xdt + \sigma dZ, \tag{5}$$

where $m(t, x)$ is a smooth and slowly varying function of x and t, which can be interpreted as the mean reversion function. The constant σ is the volatility. dZ is a Wiener process,

$$dZ = \varepsilon \sqrt{dt}, \tag{6}$$

where ε is a random drawing from a standard normal distribution with a mean zero and a standard deviation one.

Let us first show that we can reduce a more general Stochastic process

$$dx' = -\widetilde{m}(t, x')\, x'\, dt + \widetilde{\sigma}(t, x')dZ, \tag{7}$$

to the form (5). In equation (7), $\widetilde{m}(t,x')$ and $\widetilde{\sigma}(t,x')$ are two smooth and slowly varying functions of x' and t. Let us consider a general transformation,

$$x = f(t, x'). \tag{8}$$

According to Ito's lemma,

$$dx = \left[\frac{\partial f}{\partial t} - \widetilde{m}(t,x')\, x' \frac{\partial f}{\partial x'} + \frac{1}{2}\widetilde{\sigma}^2(t,x')\frac{\partial^2 f}{\partial (x')^2}\right] dt + \frac{\partial f}{\partial x'}\widetilde{\sigma}(t,x') dZ. \tag{9}$$

If we choose the transformation f such that

$$\frac{\partial f(t,x')}{\partial x'} = \frac{\sigma}{\widetilde{\sigma}(t,x')}, \tag{10}$$

where σ is a constant, representing some "average volatility" of $\widetilde{\sigma}(t,x')$, then the Stochastic differential equation for the random variable x will have the form of equation (5), with the mean reversion given by

$$m(t,x) = \widetilde{m}(t,x')\frac{x'}{x}\frac{\partial f}{\partial x'} - \frac{1}{x}\left[\frac{\partial f}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2(t,x')\frac{\partial^2 f}{\partial (x')^2}\right]. \tag{11}$$

Although the volatility σ is constant, the one-factor model represented by equations (4) and (5) is not limited to Gaussian models. Let us consider a simple example: Volatility smile,

$$\widetilde{\sigma}(t,x') = \sigma\left[\alpha_1(t) + \alpha_2(t)(x')^2\right]. \tag{12}$$

The appropriate transformation to reduce the Stochastic process for equation (12) to the form (5) is,

$$x = \frac{1}{\sqrt{\alpha_1(t)\alpha_2(t)}} \arctan\left(x'\sqrt{\frac{\alpha_2(t)}{\alpha_1(t)}}\right). \tag{13}$$

The resulting mean reversion for the random variable x is

$$m(t,x) = \frac{x'}{x}\left\{\sigma^2\alpha_2(t) + \frac{\widetilde{m}(t,x') + \frac{1}{2}\frac{d}{dt}\ln\left[\frac{\alpha_1(t)}{\alpha_2(t)}\right]}{\alpha_1(t) + \alpha_2(t)(x')^2}\right\} + \frac{1}{2}\frac{d}{dt}\ln\left[\alpha_1(t)\alpha_2(t)\right]. \tag{14}$$

Some well known term structure models are special cases of the general one-factor interest rate model represented by equations (4) and (5). For instance, the Black-Derman-Toy model (Black and Derman and Toy 1990) is defined as,

$$dr = a(t)r\,dt + \sigma(t)r\,dZ. \tag{15}$$

Under the transformation,

$$r(t) = r_0 \exp\left[\frac{\sigma(t)}{\sigma}x + \int_0^t d\tau \left(a(\tau) - \frac{\sigma^2(\tau)}{2}\right)\right], \tag{16}$$

we reduce the Black-Derman-Toy model to the form (5) with

$$m(t,x) = \frac{d}{dt}\ln\sigma(t). \tag{17}$$

For the slightly more general Black-Karasinski model (Black and Karasinski 1991),

$$d\ln r = \phi(t)\left[\ln\mu(t) - \ln r\right]dt + \sigma(t)dZ, \tag{18}$$

the appropriate transformation is

$$\ln r(t) = \frac{\sigma(t)}{\sigma}x + \ln\mu(t) - \phi_0(t), \tag{19}$$

$$\phi_0(t) = \exp\left[-\int_0^t d\tau\phi(\tau)\right]\int_0^t d\tau \exp\left[\int_0^\tau d\tau'\phi(\tau')\right]\frac{d}{d\tau}\ln\mu(\tau). \tag{20}$$

The Black-Karasinski model is reduced to the form (5) with

$$m(t,x) = \frac{d}{dt}\ln\sigma(t) + \phi(t). \tag{21}$$

To calculate the present value of future cash flows, we introduce a Green's function $G(t,x|t',x')$, constrained to $t \le t'$. The backward induction of the cash flows can be formally expressed as

$$C(t,x) = \int_{-\infty}^{\infty} dx'\,G(t,x|t',x')C(t',x'), \tag{22}$$

where $C(t,x)$ is the cash flow at time t and in a state in which the interest rate is determined by random variable value x. In Appendix A, we show that the Green's function satisfies the following generalized Black-Scholes equation,

$$-\frac{\partial}{\partial t}G(t,x|t',x') = \hat{H}(t,x)G(t,x|t',x'), \tag{23}$$

where we have introduced a short hand notation

$$\hat{H}(t,x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - m(t,x)x\frac{\partial}{\partial x} - r(t,x). \tag{24}$$

The Green's function is subject to boundary condition,

$$\lim_{t\to t', t<t'} G(t,x|t',x') = \delta(x-x'). \tag{25}$$

For multi-factor models, the term structure is driven by more than one random factors. To simplify notations, let us consider two-factor models. When we choose the short rate as the starting point, we can write

$$r = r(t,x_1,x_2). \tag{26}$$

Without losing generality, we can assume that the Stochastic process governing the two random variables is

$$dx_1 = a_1(t,x_1,x_2)dt + \sigma_1 dZ_1, \tag{27}$$

$$dx_2 = a_2(t,x_1,x_2)dt + \sigma_2 dZ_2, \tag{28}$$

where σ_1 and σ_2 are two constants. dZ_1 and dZ_2 are two independent Wiener processes.

First let us show that we can reduce a general two-factor model to the above form by an appropriate transformation. A general Stochastic process would be

$$dx_1' = \widetilde{a}_1(t,x_1',x_2')dt + \sigma_{11}(t,x_1',x_2')dZ_1 + \sigma_{12}(t,x_1',x_2')dZ_2, \tag{29}$$

$$dx_2' = \widetilde{a}_2(t,x_1',x_2')dt + \sigma_{21}(t,x_1',x_2')dZ_1 + \sigma_{22}(t,x_1',x_2')dZ_2. \tag{30}$$

Let us consider transformation

$$x_1 = f_1(t,x_1',x_2'), \tag{31}$$

$$x_2 = f_2(t,x_1',x_2'). \tag{32}$$

We demand the transformation to satisfy

$$\begin{pmatrix} \frac{\partial f_1(t,x_1',x_2')}{\partial x_1'} & \frac{\partial f_1(t,x_1',x_2')}{\partial x_2'} \\ \frac{\partial f_2(t,x_1',x_2')}{\partial x_1'} & \frac{\partial f_2(t,x_1',x_2')}{\partial x_2'} \end{pmatrix} \begin{pmatrix} \sigma_{11}(t,x_1',x_2') & \sigma_{12}(t,x_1',x_2') \\ \sigma_{21}(t,x_1',x_2') & \sigma_{22}(t,x_1',x_2') \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{33}$$

The Stochastic equations (29) and (30) are reduced to the form (27) and (28), with coefficients given by

$$\begin{aligned} a_1(t,x_1,x_2) &= \frac{\partial f_1}{\partial t} + \tilde{a}_1 \frac{\partial f_1}{\partial x_1'} + \tilde{a}_2 \frac{\partial f_1}{\partial x_2'} + \frac{\sigma_{11}^2+\sigma_{12}^2}{2}\frac{\partial^2 f_1}{\partial (x_1')^2} + \frac{\sigma_{21}^2+\sigma_{22}^2}{2}\frac{\partial^2 f_1}{\partial (x_2')^2} \\ &\quad + (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})\frac{\partial^2 f_1}{\partial x_1' \partial x_2'}, \end{aligned} \tag{34}$$

$$\begin{aligned} a_2(t,x_1,x_2) &= \frac{\partial f_2}{\partial t} + \tilde{a}_1 \frac{\partial f_2}{\partial x_1'} + \tilde{a}_2 \frac{\partial f_2}{\partial x_2'} + \frac{\sigma_{11}^2+\sigma_{12}^2}{2}\frac{\partial^2 f_2}{\partial (x_1')^2} + \frac{\sigma_{21}^2+\sigma_{22}^2}{2}\frac{\partial^2 f_2}{\partial (x_2')^2} \\ &\quad + (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})\frac{\partial^2 f_2}{\partial x_1' \partial x_2'}. \end{aligned} \tag{35}$$

The backward induction of the cash flows is formally carried out in terms of Green's function, in complete analogy to the one-factor equation (22). The generalized Black-Scholes equation for the Green's function is

$$-\frac{\partial}{\partial t} G(t,x_1,x_2|t',x_1',x_2') = \hat{H}(t,x_1,x_2) G(t,x_1,x_2|t',x_1',x_2'), \tag{36}$$

$$\hat{H}(t,x_1,x_2) = \frac{\sigma_1^2}{2}\frac{\partial^2}{\partial x_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2}{\partial x_2^2} + a_1(t,x_1,x_2)\frac{\partial}{\partial x_1} + a_2(t,x_1,x_2)\frac{\partial}{\partial x_2} - r(t,x_1,x_2). \tag{37}$$

The boundary condition is,

$$\lim_{t\to t', t<t'} G(t,x_1,x_2|t',x_1',x_2') = \delta(x_1-x_1')\,\delta(x_2-x_2'). \tag{38}$$

The price of an European option is expressed in terms of the Green's function, similar to equation (22). The expression for the American option price will be derived in the next section.

It is interesting and useful to point out that equations (36) and (37) are formally identical to the Schrödinger equation studied in quantum statistical mechanics. We list the correspondence in Table I.

II. EARLY EXERCISE PREMIUM OF AMERICAN OPTIONS

The price of an American option can be expressed as the sum of the price of the corresponding European option plus an additional term representing the early exercise premium. The early exercise premium is generally an integral over the critical boundary on which it is advantageous to exercise the American option immediately. In this section, we present the early exercise premium representation for American options in general Markovian interest rate models. Without losing generality, we present our results for the two-factor models.

Let us define a general American option: If the option is exercised at time t, where t is no later than the maturity time T, the option payoff is

$$C(t, x_1, x_2) = c(t, x_1, x_2)\, \theta(c(t, x_1, x_2)). \tag{39}$$

The step function $\theta(c)$ is defined as,

$$\theta(c) = \begin{cases} 1, & c > 0 \\ \frac{1}{2}, & c = 0 \\ 0, & c < 0. \end{cases} \tag{40}$$

Intuitively, one can think of the step function as a limiting case of the normal Gaussian integral $N(x)$,

$$\theta(c) = \lim_{w \to 0} N\left(\frac{c}{w}\right), \tag{41}$$

$$N(x) = \int_{-\infty}^{x} \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right). \tag{42}$$

We assume that the function $c(t, x_1, x_2)$ in equation (39) is a smooth and slowly varying function of time and random variables. Most options satisfy this condition, except for compound options for which the underlying security itself is an option close to maturity. For an European option whose payoff at maturity T is given by (39), the option value at an earlier time t and in the state (x_1, x_2) is,

$$P(t, x_1, x_2) = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2\, G(t, x_1, x_2 | T, y_1, y_2) C(T, y_1, y_2). \tag{43}$$

We recall that the Green's function $G(t, x_1, x_2|T, y_1, y_2)$ satisfies differential equation (36). The price of an American option is,

$$P_a(t, x_1, x_2) = P(t, x_1, x_2) + E(t, x_1, x_2). \tag{44}$$

In Appendix A, we prove that the early exercise premium is

$$E(t, x_1, x_2) = \int_t^T d\tau \int\int_{c(\tau,y_1,y_2)\geq P_a(\tau,y_1,y_2)} dy_1 dy_2\, G(t, x_1, x_2|\tau, y_1, y_2) A(\tau, y_1, y_2), \tag{45}$$

$$A(\tau, y_1, y_2) = -\left[\frac{\partial}{\partial \tau} + \hat{H}(\tau, y_1, y_2)\right] c(\tau, y_1, y_2), \tag{46}$$

where $\hat{H}(\tau, y_1, y_2)$ is given by (37). We point out that $A(\tau, y_1, y_2)$ is solely dependent on the underlying security of the option. Obviously, early exercise is advantageous only if $A(\tau, y_1, y_2) > 0$. For the American call option on a stock without dividend, this is negative. Therefore, as expected, early exercise should not happen for the American call stock option.

The early exercise premium depends on the critical boundary separating the American and European regions. The boundary equation is

$$c(\tau, y_1, y_2) - P_a(\tau, y_1, y_2) = 0. \tag{47}$$

The critical boundary needs to be calculated iteratively from the maturity backward. Fortunately, the critical boundary near maturity in Markovian models can be calculated analytically. In one-factor Markovian models, the critical boundary is a line determined by $c(t, x^*(t)) = P_a(t, x^*(t))$. Close to the maturity, the critical line has a *universal* form. We prove in Appendix B, as $t \to T$,

$$x^*(t) = x^*(T) + \sigma \sqrt{-(T-t)\left[\ln\left(\frac{T-t}{\tau_0}\right) + \frac{2}{\ln\left(\frac{T-t}{\tau_0}\right)}\right]}, \tag{48}$$

$$\tau_0 = \frac{1}{8\pi}\left[\frac{\sigma}{A(T,x)}\frac{\partial c(T,x)}{\partial x}\right]^2_{x=x^*(T)}. \tag{49}$$

Generally, the early exercise premium (45) has discontinuous second and higher order derivatives with respect to the random variables across the boundary.

Let us apply the early exercise premium expression to American Swaptions that swap a floater with a fixed coupon bond. The price at time t for the fixed rate coupon bearing bond of face value of one dollar with maturity T and coupon rate c_0 is, assuming semiannual coupon payment,

$$B(t, x_1, x_2|t+T) = c_0 \sum_{i=1}^{2T} Z(t, x_1, x_2|t+i/2) + Z(t, x_1, x_2|t+T), \tag{50}$$

where $Z(t, x_1, x_2|t+T)$ is the price of a zero coupon bond of maturity T. It is given by

$$Z(t, x_1, x_2|t+T) = \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dx_2' G(t, x_1, x_2|t+T, x_1', x_2'). \tag{51}$$

When the Swaption is exercised, its payoff is given by equation (39) with

$$c(t, x_1, x_2) = 1 - B(t, x_1, x_2|t+T). \tag{52}$$

It is straightforward to verify, from equation (46),

$$A(t, x_1, x_2) = r(t, x_1, x_2) + c_0 \sum_{i=1}^{2T} \left[\frac{\partial}{\partial \tau} Z(t, x_1, x_2|t+\tau) \right]_{\tau=i/2} + \frac{\partial}{\partial T} Z(t, x_1, x_2|t+T). \tag{53}$$

The expression for the early exercise premium (45) is one of the main results of this paper. In the next section, we shall decompose the cash flows of European options into fast and slowly varying components and derive analytic expressions for backward inducting these components. Together with early exercise premium equation (45), we shall have analytic expressions for backward inducting the cash flows of both European and American options for a finite time period with controlled precision.

III. ANALYTICAL BACKWARD INDUCTION OF OPTION CASH FLOWS

Our goal is to derive analytic expressions to backward induct option cash flows for a finite time interval with controlled precision. To simplify notations, our presentation will be focused on the general one-factor model represented by equations (4) and (5). It will be clear to the readers that the treatment for general multi-factor models is completely analogous, as long as the short interest rate is history independent.

The early exercise premium derived in the last section is the first step needed to achieve our goal. The next step is to derive the analytic expressions for cash flow induction. Specifically, given the option cash flow function $C(t_2, x)$ at time t_2, we want to derive an analytic expression for the cash flow function $C(t_1, x)$ at the time $t_1 = t_2 - \delta t$ for finite δt. The maximum allowed value for δt will be chosen to meet precision requirement. If the option cash flow function $C(t, x)$ were slowly varying, that is, if

$$\left| \delta x \, \frac{\partial}{\partial x} C^{(n)}(t, x) \right| \ll \left| C^{(n)}(t, x) \right|, \quad \text{for } |\delta x| \ll 1, \tag{54}$$

were satisfied for derivatives $C^{(n)}(t, x) = \partial^n C(t, x)/\partial x^n$ at least to the second order, i.e. $n \leq 2$, we would be able to derive an approximate solution directly from the generalized Black-Scholes equation (23). If the cash flow function representing the option payoff upon exercising satisfied condition (54), all backward inducted cash flows from equation (22) would satisfy (54). Unfortunately, the option cash flow function violates condition (54). Without loss of generality, we can express the option cash flow function at maturity T as

$$C(T, x) = c(T, x)\theta(c(T, x)) = c(T, x)\theta(x - x_T^*), \tag{55}$$

where x_T^* is the critical point at the maturity, $\theta(x)$ is the step function defined in (40). We have adopted the convention that $x > x_T^*$ is the American region in which the option should be exercised immediately. The origin of the fast varying components in the cash flow function is the step function in the exercise generated cash flows in equation (55).

The necessity of separating out the fast varying components from the cash flow function can be understood from a simple example. Let us consider the explicit finite difference method (Hull 1997) for a pure diffusion problem, i.e. equation (23) with $m(t, x) = 0$ and $r(t, x) = 0$. The goal is to find the distribution from an initial one after propagating a finite time period. Let us assume that we choose equally spaced points to implement the finite difference scheme, and δt and δx are the spacing along t and x axis respectively. Let us denote

$$\xi(k) = 1 - \frac{2\sigma^2 \, \delta t}{\delta x^2} \sin^2\left(\frac{k \, \delta x}{2}\right), \tag{56}$$

where k is a real wave number in the decomposition of the initial distribution. The von Neumann stability condition for the finite difference schemes (Press, Teukolsky, Vetterling and Flannery 1992) requires the amplitude factor $|\xi(k)| \leq 1$. For fast varying initial distribution, the wave number k can take any value. In this case, the von Neumann stability condition requires $\sigma^2\,\delta t \leq \delta x^2$. Other finite difference schemes, such as the implicit finite difference method and Crank-Nickolson scheme, can eliminate this constraint and achieve unconditional numerical stability. One should realize that any tree construction is essentially a finite difference scheme. Although some schemes have faster convergence than others, the basic limitation is the same: If x_c is the scale over which the initial distribution changes significantly, then one is forced to choose $\delta x < x_c$. For a diffusion time period T, the number of numerical steps would be of order $\sigma^2 T/x_c^2$. For fast varying initial distribution, $x_c \ll 1$, leading to extremely large number of numerical steps. However, if the initial distribution satisfies condition (54), then the largest possible wave number in the amplitude factor (56) is limited to $k \sim 1/x_c \sim 1$. The amplitude factor is reduced to $\xi(k) \simeq 1 - k^2\sigma^2\,\delta t/2$ for $\delta x < 1$. Not only the number of numerical steps is greatly reduced, but also the von Neumann stability condition is not a concern anymore for $\sigma^2\,\delta t \ll 1$.

We introduce two functions, $F_0(t,x)$ and $F_1(t,x)$, to capture the fast varying components in the option cash flow function. The differential equation for $F_0(t,x)$ is

$$-\frac{\partial}{\partial t}F_0(t,x) = \left[\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - m_0(t)x\frac{\partial}{\partial x}\right]F_0(t,x), \tag{57}$$

$$\lim_{t\to T, t<T} F_0(t,x) = \theta(x - x_T^*), \tag{58}$$

where $m_0(t)$ is independent of x, representing some "average" mean reversion at time t. We shall assume that $|m(t,x) - m_0(t)| < \sigma^2$. $F_0(t,x)$ can be derived analytically.

$$F_0(t,x) = N\left(\frac{x - \lambda_1(t,T)x_T^*}{\sigma\sqrt{(T-t)\lambda_2(t,T)}}\right), \tag{59}$$

where $N(x)$ is the normal Gaussian integral defined by equation (42). We have introduced two time dependent functions, which only depend on the mean reversion curve $m_0(t)$,

$$\lambda_1(t,t') = \exp\left[\int_t^{t'} d\tau\, m_0(\tau)\right], \tag{60}$$

$$\lambda_2(t,t') = \frac{\lambda_1^2(t,t')}{t'-t}\int_t^{t'}\frac{d\tau}{\lambda_1^2(\tau,t')}. \tag{61}$$

We note that both $\lambda_1(t,t') = 1$ and $\lambda_2(t,t') = 1$ as $t \to t'$. The function $F_1(t,x)$ is calculated from $F_0(t,x)$,

$$F_1(t,x) = \frac{\partial}{\partial x} F_0(t,x). \tag{62}$$

The value of an European option with cash flows at maturity given by (55) can be expressed as

$$P(t,x) = F_0(t,x)Q(t,x) + F_1(t,x)R(t,x), \tag{63}$$

where $Q(t,x)$ and $R(t,x)$ are two *slowly varying* functions. The option specific information is solely contained in these two functions. They satisfy the following equations

$$-\frac{\partial}{\partial t}Q(t,x) = \left[\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - m(t,x)x\frac{\partial}{\partial x} - r(t,x)\right]Q(t,x), \tag{64}$$

$$-\frac{\partial}{\partial t}R(t,x) = \left\{\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left[b(t)x_T^* - \widetilde{m}(t,x)x\right]\frac{\partial}{\partial x} - \widetilde{r}(t,x)\right\}R(t,x) + J(t,x). \tag{65}$$

We have introduced the following convenient notations,

$$b(t) = \frac{\lambda_1(t,T)}{(T-t)\lambda_2(t,T)}, \tag{66}$$

$$\widetilde{m}(t,x) = m(t,x) + \frac{1}{(T-t)\lambda_2(t,T)}, \tag{67}$$

$$\widetilde{r}(t,x) = r(t,x) - m_0(t) - \frac{\left[m(t,x) - m_0(t)\right]\left[x - \lambda_1(t,T)x_T^*\right]x}{\sigma^2(T-t)\lambda_2(t,T)}, \tag{68}$$

$$J(t,x) = \left\{\sigma^2\frac{\partial}{\partial x} - \left[m(t,x) - m_0(t)\right]x\right\}Q(t,x). \tag{69}$$

The boundary conditions are

$$\lim_{t\to T, t\le T} Q(t,x) = c(T,x), \tag{70}$$

$$\lim_{t\to T, t\le T} R(t,x) = 0. \tag{71}$$

The presence of $(T-t)$ in the denominator of $b(t)$, $\widetilde{m}(t,x)$ and $\widetilde{r}(t,x)$ is harmless. Because $\lim_{t\to T} R(t,x) = 0$, we see that $R(t,x)/(T-t)$ is no more than the first order derivative with respect to time in the limit $t \to T$. Since $Q(t,x)$ and $R(t,x)$ are two slowly varying functions of time and random variable x, we can derive approximate solutions for them from an expansion in the small parameter $\sigma^2(t_2 - t_1)$ for the time period (t_1, t_2). The results are presented in the equations (B7), (B8), (B16), and (B17) in Appendix B.

IV. AMERICAN PUT OPTION ON A STOCK WITHOUT DIVIDEND

In this section, we derive analytic expressions to approximate the boundary of critical stock prices for an American put option. The early exercise premium will be an integral over the critical boundary. Denoting the stock price by S, the strike price by K, and the risk free rate by r, the Black-Scholes equation for the price $P(t,S)$ of an European option in the risk neutral world is

$$\frac{\partial P(t,S)}{\partial t} + rS\frac{\partial P(t,S)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P(t,S)}{\partial S^2} = rP(t,S). \tag{72}$$

Under the variable transformation

$$S(t,x) = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t - x\right], \tag{73}$$

where S_0 is the stock price at $t=0$, the Black-Scholes equation is reduced to

$$\left[\frac{\partial}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - r\right] P(t,x) = 0. \tag{74}$$

For an American put option, the exercise generated cash flows are

$$[K - S(t,x)]\,\theta\left(K - S(t,x)\right), \tag{75}$$

where the step function $\theta(c)$ is defined in equation (40). At the maturity time T, the critical stock price is $S(T, x^*(T))$ with

$$x^*(T) = \ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T. \tag{76}$$

The option payoff is positive at $x > x^*(T)$. For the early exercise premium given by equation (45), it is easy to verify,

$$A(t,x) = -\left[\frac{\partial}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - r\right][K - S(t,x)] = rK. \tag{77}$$

The early exercise premium representation for the American put option is

$$P_a(t,x) = P(t,x) + rK\int_t^T d\tau \exp[-(\tau - t)r] N\left(\frac{x - x^*(\tau)}{\sigma\sqrt{\tau - t}}\right), \tag{78}$$

where $P(t, x)$ is the price of the corresponding European option. At time $t < T$, the critical point $x^*(t)$ is determined by

$$K - S(t, x^*(t)) = P(t, x^*(t)) + rK \int_t^T d\tau \exp[-(\tau - t)r] N\left(\frac{x^*(t) - x^*(\tau)}{\sigma\sqrt{\tau - t}}\right). \tag{79}$$

Although the exact analytic solution for $x^*(t)$ is not available, we can derive the asymptotic forms at both close to and far away from the maturity.

The equation (74) is formally a special case of the one-factor interest rate model represented by equations (23) and (24). The universal critical boundary near maturity given by equation (48) is also valid in this case. In particular, the characteristic time scale in equation (49) becomes

$$\tau_0 = \frac{\sigma^2}{8\pi r^2}. \tag{80}$$

The critical boundary $x^*(t)$ can be expressed in terms of a two dimensional function $g(u, v)$,

$$x^*(t) = x^*(T) - \left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{\tau_0}\, g\left(\frac{T - t}{\tau_0}, \sigma\sqrt{\tau_0}\right). \tag{81}$$

The boundary equation determining $g(u, v)$ is

$$\begin{aligned} &\exp\left(-\frac{uv}{\sqrt{8\pi}}\right) N\left(\frac{v\sqrt{u}}{2} - \sqrt{\frac{u}{8\pi}} + \frac{g(u,v)}{\sqrt{u}}\right) + \exp\left[-v\, g(u,v)\right] N\left(\frac{v\sqrt{u}}{2} + \sqrt{\frac{u}{8\pi}} - \frac{g(u,v)}{\sqrt{u}}\right) \\ &+ \frac{uv}{\sqrt{8\pi}} \int_0^1 d\lambda \exp\left(-\frac{uv\lambda}{\sqrt{8\pi}}\right) N\left(\frac{v\sqrt{u\lambda}}{2} - \sqrt{\frac{u\lambda}{8\pi}} + \frac{g(u,v) - g((1-\lambda)u, v)}{\sqrt{u\lambda}}\right) = 1, \end{aligned} \tag{82}$$

where $N(x)$ is the Gaussian integral defined in equation (42). The physical meaning of the two variables u and v is

$$u = \frac{T - t}{\tau_0}, \qquad v = \sigma\sqrt{\tau_0} = \frac{\sigma^2}{r\sqrt{8\pi}}. \tag{83}$$

The asymptotic form for the critical boundary at $u \gg 1$ can be derived from equation (82) with straightforward algebra. Combined with the asymptotic form at $u \ll 1$, we find

$$g(u, v) = \begin{cases} \sqrt{-u\left[\ln(u) + \frac{2}{\ln(u)}\right]}, & u \ll 1 \\ \frac{1}{v}\ln(1 + v\sqrt{2\pi}) - \frac{g_\infty(v)}{u\sqrt{2\pi}} \exp\left[-\frac{u}{16\pi}(1 + v\sqrt{2\pi})^2\right], & u \gg 1. \end{cases} \tag{84}$$

The constant $g_\infty(v)$ depends on the crossover behavior of the critical boundary and it not available analytically.

For practical applications, it suffices to obtain an approximate analytic expression for $g(u,v)$ with sufficient accuracy. For a given value v, we have found the following analytic expressions,

$$g(u,v) = \begin{cases} \sqrt{-u\left[\ln(u) + \frac{2}{\ln(u)}\right]} + a_1(v)u + a_2(v)u^2, & u < u_1(v) \\ \exp\left[-\frac{u}{16\pi}(1+v\sqrt{2\pi})^2\right] \sum_{n=1}^{n=12} \frac{b_n(v)}{(1+b_0(v)\sqrt{u})^n}, & u_1(v) \le u < u_2(v), \\ \frac{1}{v}\ln(1+v\sqrt{2\pi}) & u \ge u_2(v). \end{cases} \tag{85}$$

We determine the coefficients $a_n(v)$, $u_n(v)$, and $b_n(v)$ by fitting the numerical solution of equation (82) to expression (85). The dividing point $u_1(v)$ is chosen such that $g(u_1(v),v) = 0.3\, g(\infty, v)$. For wide ranges of v, the relative error of the worst fitting is better than 10^{-3}. Our strategy is to fit the coefficients for a set of properly selected v values. For values other than the selected ones, we obtain $g(u,v)$ via polynomial interpolation of the boundary functions of the pre-selected neighboring v values. Specifically, for $v_1 < v_2 < v < v_3 < v_4$ with function values $g(u,v_i)$ from $i=1$ to $i=4$ calculated from (85), the function value at the desired v is

$$g(u,v) = \text{Polint}\left(g(u,v_1), g(u,v_2), g(u,v_3), g(u,v_4)\right), \tag{86}$$

where Polint is the standard function of third order polynomial interpolation. In terms of the critical boundary, the American put option price is calculated directly from equation (78). Setting $t=0$ and $x=0$ in equation (78), we obtain for the American put price P_a,

$$P_a = P + rT\,K\,g_p\left(x^*(T), \frac{T}{\tau_0}, \sigma\sqrt{\tau_0}\right), \tag{87}$$

$$g_p(\alpha_1,\alpha_2,\alpha_3) = \int_0^1 d\lambda\, \exp\left(-\lambda\frac{\alpha_2\alpha_3}{\sqrt{8\pi}}\right) \times N\left(\frac{\sqrt{\alpha_2}}{2}\left(\frac{1}{\sqrt{2\pi}} - \alpha_3\right)\frac{1-\lambda}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda\alpha_2}}\left[\frac{\alpha_1}{\alpha_3} + g\left(\alpha_2(1-\lambda), \alpha_3\right)\right]\right), \tag{88}$$

where P is the Black-Scholes European option price, and $x^*(T)$ and τ_0 are given by equations (76) and (80) respectively.

We have tested the put option price calculated from equation (86) for a wide range of possible parameters, from small to 100% volatilities and maturity up to five years. The relative precision is better than 10^{-5} for all cases. For longer maturity and higher volatility, the accuracy for the put option price should be preserved since the boundary function $g(u, v)$ is already well approximated by its asymptotic form at $u \gg 1$..

From the critical boundary (86), we can calculate the maximum strike price K_m for a put option. If the current price of the stock is S_0, the maximum allowed strike price for a put option of maturity T is determined by $x^*(t = 0) = 0$. The result is

$$K_m(T) = S_0 \exp\left[\sigma\sqrt{\tau_0}\, g\left(\frac{T}{\tau_0}, \sigma\sqrt{\tau_0}\right)\right]. \tag{89}$$

Any put option with strike price higher than K_m should be exercised immediately, in the Black-Scholes model. For a perpetual put option, i.e. $T = \infty$, we recover the previously known result $K_m(\infty) = S_0(1 + \sigma^2/(2r))$.

The calculation of the put option price from equations (88) and (86) is fast and accurate. However, for practical applications, an even simpler solution exists. In typical applications with option maturity $T \sim 1$ and $r \sim 5\%$, we observe that $g(u, v)$ for large u is needed only for small volatilities, i.e. for small v. Thus, an expansion in v around $v = 0$ forms a good starting point for practical applications,

$$g(u, v) = g_0(u) + uvg_1(u). \tag{90}$$

The equations that determine the two universal functions $g_0(u)$ and $g_1(u)$, as well as the coefficients for their approximate analytic expressions, are included in Appendix C. The advantage of equation (90) is that it is specified by a small number of parameters. In Figure 2, we show the theoretical boundary (90) compared with the results calculated from the CRR binomial tree (Cox, Ross, and Rubinstein, 1979). The agreement is excellent. The put option prices calculated from the boundary (90) are shown in Table III. They have excellent precision. Only for options with extremely large volatilities and long maturities, we have to resort to the more accurate boundary (86).

V. NUMERICAL PROCEDURE FOR INTEREST RATE MODELS

Equations (44), (45) and (63) outline a new powerful approach to calculate efficiently the price of both European and American options for general multi-factor Markovian interest rate models. In this approach, the option time horizon is divided into a relatively small number of time intervals. For each time interval, the option cash flows are backward inducted via analytic expressions. The numerical implementation amounts to iterate the analytic expressions of one time interval. The analytic expressions for one time interval are derived from an expansion in the small parameter $\sigma^2 \cdot \delta t$, where δt is the width of the time interval. We have derived the expansion solution for one-factor models in Appendix B. The omitted terms are of order $(\sigma^2 \cdot \delta t)^2$. For typical volatilities of order 30% in the interest rate models, one can achieve accuracy of better than 10^{-2} with $\delta t \leq 1$ for the analytic expressions.

We conclude this paper by summarizing the steps for numerical implementation.

1. Given a general Markovian interest rate model, first transform it to a form similar to equations (5) or (27) and (28).

2. For a given American option, derive the expression (46) for the early exercise premium from the underlying security of the option.

3. Decompose the European option into fast and slowly varying components and derive the analytic expressions for the fast varying components.

4. Derive the differential equations for the slowly varying components and derive the expansion solution for a finite time interval. At the same time, derive the iteration for the critical boundary.

5. Numerically iterate the analytic expressions for the backward induction of one time interval so that the backward induction is carried out for the entire time period from the option maturity to the present time.

VI. CONCLUSION

This paper presents the formulation of general multi-factor interest models for which the short rate is history independent. We argue that this class of models contain all realistic market models. We have derived the early exercise premium representation for American options whose payoff is history independent. We have shown how to separate out the fast varying components from the option cash flows and calculate them analytically. The slowly varying components are calculated from controlled expansion. We have derived explicit analytic expressions for backward inducting option cash flows for the one-factor models. We have shown that the critical boundary for the American options in the one-factor models has a universal form near the maturity. For an American put stock option, we have derived the exact asymptotic forms for the critical boundary close to and far away from the maturity. We have obtained approximate analytic expressions for the entire critical boundary of the American put stock option. The early exercise premium calculated as an integral over the critical boundary achieves the accuracy that is only matched by CRR binomial tree with over $100,000$ time steps.

APPENDIX A: GREEN'S FUNCTION AND EARLY EXERCISE PREMIUM REPRESENTATION

In this section, we present detailed derivation of the backward cash flow equation and the early exercise premium representation for a general American option. To simply notations, we present our derivation for the two-factor models, though the derivation is valid for any number of random factors as long as the short rate is history independent.

Let us consider a time period from t to T. The Stochastic process is governed by equations (27) and (28). We divide the time period into N small intervals,

$$t < t_1 < t_2 \cdots < t_{N-1} < t_N = T, \quad t_{i+1} - t_i = \epsilon. \tag{A1}$$

For a small time interval (t_i, t_{i+1}), the probability from a state (x_1, x_2) at time t_i to a state

(x_1', x_2') at time t_{i+1} is

$$P(t_i, x_1, x_2|t_i + \epsilon, x_1', x_2') = P(t_i, x_1|t_i + \epsilon, x_1'; \sigma_1, a_1)P(t_i, x_2|t_i + \epsilon, x_2'; \sigma_2, a_2), \tag{A2}$$

$$P(t_i, x_1|t_i + \epsilon, x_1'; \sigma_1, a_1) = \frac{1}{\sigma_1\sqrt{2\pi\epsilon}} \exp\left[-\frac{(x_1' - x_1 - a_1(t_i, x_1, x_2)\epsilon)^2}{2\sigma_1^2\epsilon}\right]. \tag{A3}$$

The Green's function for the time interval $(t_i, t_i + \epsilon)$ is, in the risk neutral world,

$$G(t_i, x_1, x_2|t_i + \epsilon, x_1', x_2') = P(t_i, x_1, x_2|t_i + \epsilon, x_1', x_2') \exp\left[-r(t_i, x_1, x_2)\epsilon\right]. \tag{A4}$$

Let us introduce the following short hand notions,

$$\mathbf{x}_i = \mathbf{x}(t_i) = (x_1(t_i), x_2(t_i)), \qquad \int d\mathbf{x} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2. \tag{A5}$$

For the finite period (t, T), the Green's function is defined by

$$G(t, \mathbf{x}|T, \mathbf{x}') = \lim_{\epsilon\to 0, N\to\infty} G_N(t, \mathbf{x}|T, \mathbf{x}'), \tag{A6}$$

$$G_N(t, \mathbf{x}|T, \mathbf{x}') = \int d\mathbf{y}_1 G\left(t, \mathbf{x}|t_1, \mathbf{y}_1\right) G\left(t_1, \mathbf{y}_1|t_2, \mathbf{y}_2\right) \cdots$$

$$\times \int d\mathbf{y}_{N-1} G\left(t_{N-2}, \mathbf{y}_{N-2}|t_{N-1}, \mathbf{y}_{N-1}\right) G\left(t_{N-1}, \mathbf{y}_{N-1}|T, \mathbf{x}'\right). \tag{A7}$$

The expression (A6) is the path integral representation of the Green's function. The path integral, introduced by Feymann, has been studied extensively in physics (Feynman and Hibbs 1965). The application of path integral to finance has been discussed by Dash (Dash 1986, 1989). It is interesting to note that the pricing formula of Geske and Johnson (Geske and Johnson 1984) for American put stock option as the sum of an infinite series is in essence an explicit expression of the equation (A7). Following the standard treatment of the path integral, one derives the differential equation (36) for the Green's function (A6).

Next we turn to the early exercise premium representation of American options. At time t_i before maturity, let us denote by $M(t_i, \mathbf{y}_i)$ the value of the option should it not be exercised at time t_i. On the other hand, if the option were exercised, the generated cash flows would be $c(t_i, \mathbf{y}_i)$ introduced in (39). The true value of the American option is

$$P_a(t_i, \mathbf{y}_i) = M(t_i, \mathbf{y}_i)\theta\left(M(t_i, \mathbf{y}_i) - c(t_i, \mathbf{y}_i)\right) + c(t_i, \mathbf{y}_i)\theta\left(c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)\right), \tag{A8}$$

where $\theta(c)$ is the step function defined in equation (40). The last expression can be written in an alternative form

$$P_a(t_i, \mathbf{y}_i) = M(t_i, \mathbf{y}_i) + [c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)]\, \theta\left(c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)\right). \tag{A9}$$

The first term is equal to the value at time t_i of all future cash flows of the option. The second term represents the early exercise premium at time t_i. Backward inducting to time t_{i-1}, the cash flow function is

$$M(t_{i-1}, \mathbf{y}_{i-1}) = \int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1} | t_i, \mathbf{y}_i\right) P_a(t_i, \mathbf{y}_i). \tag{A10}$$

Explicitly, we can write the last expression as

$$\begin{aligned} M(t_{i-1}, \mathbf{y}_{i-1}) = &\int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1} | t_i, \mathbf{y}_i\right) M(t_i, \mathbf{y}_i) \\ &+ \int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1} | t_i, \mathbf{y}_i\right) [c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)]\, \theta\left(c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)\right). \end{aligned} \tag{A11}$$

The equations (A9) and (A10) can be iterated. Using the propagation property of the Green's function, for $t_i < t_k < t_j$,

$$G\left(t_i, \mathbf{y}_i | t_j, \mathbf{y}_j\right) = \int d\mathbf{y}_k G\left(t_i, \mathbf{y}_i | t_k, \mathbf{y}_k\right) G\left(t_k, \mathbf{y}_k | t_j, \mathbf{y}_j\right), \tag{A12}$$

we obtain from iteration,

$$M(t, \mathbf{x}) = P(t, \mathbf{x}) + \sum_{i=1}^{N} \int d\mathbf{y}_i G\left(t, \mathbf{x} | t_i, \mathbf{y}_i\right) [c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)]\, \theta\left(c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i)\right), \tag{A13}$$

where $P(t, \mathbf{x})$ is the value of the corresponding European option with the same payoff at maturity. The critical boundary $\mathbf{y}^*$ is determined by

$$c(t_i, \mathbf{y}_i^*) - M(t_i, \mathbf{y}_i^*) = 0. \tag{A14}$$

The boundary separates the American and European regions. In the American region, $c(t_i, \mathbf{y}_i^*) > M(t_i, \mathbf{y}_i^*)$, the option should be exercised immediately. If we assume that the boundary is smooth in the continuum time limit $N \to \infty$, then equation (A13) implies that in the American region,

$$\lim_{N\to\infty} c(t_i, \mathbf{y}_i) - M(t_i, \mathbf{y}_i) = A(t_i, \mathbf{y}_i)\epsilon, \tag{A15}$$

where the function $A(t_i, \mathbf{y}_i)$ has a finite positive limit as $N \to \infty$. Since in the European region, including the boundary, $P_a(t, \mathbf{x}) = M(t, \mathbf{x})$, the price of the American option in the continuum time limit becomes

$$P_a(t, \mathbf{x}) = P(t, \mathbf{x}) + \int_t^T d\tau \int d\mathbf{y} G\left(t, \mathbf{x}|\tau, \mathbf{y}\right) A(\tau, \mathbf{y}) \theta\left(c(\tau, \mathbf{y}) - P_a(\tau, \mathbf{y})\right). \tag{A16}$$

The remaining task is to determine the function $A(\tau, \mathbf{y})$. In the following, we prove

$$A(\tau, \mathbf{y}) = -\left[\frac{\partial}{\partial \tau} + \hat{H}(\tau, \mathbf{y})\right] c(\tau, \mathbf{y}), \tag{A17}$$

where $\hat{H}(\tau, \mathbf{y})$ is defined by (37). The equation (A9) can be written in another form,

$$P_a(t_i, \mathbf{y}_i) = c(t_i, \mathbf{y}_i) + \left[M(t_i, \mathbf{y}_i) - c(t_i, \mathbf{y}_i)\right] \theta\left(M(t_i, \mathbf{y}_i) - c(t_i, \mathbf{y}_i)\right). \tag{A18}$$

The equation corresponding to (A10) becomes

$$\begin{aligned} M(t_{i-1}, \mathbf{y}_{i-1}) = &\int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1}|t_i, \mathbf{y}_i\right) c(t_i, \mathbf{y}_i) \\ &+ \int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1}|t_i, \mathbf{y}_i\right) \left[M(t_i, \mathbf{y}_i) - c(t_i, \mathbf{y}_i)\right] \theta\left(M(t_i, \mathbf{y}_i) - c(t_i, \mathbf{y}_i)\right). \end{aligned} \tag{A19}$$

Since the function $c(t, \mathbf{y})$ is smooth and differentiable, we can verify

$$\int d\mathbf{y}_i G\left(t_{i-1}, \mathbf{y}_{i-1}|t_i, \mathbf{y}_i\right) c(t_i, \mathbf{y}_i) = \left[1 + \epsilon \hat{H}\left(t_{i-1}, \mathbf{y}_{i-1}\right)\right] c(t_i, \mathbf{y}_{i-1}). \tag{A20}$$

The second term on the right hand side of (A19) decreases exponentially away from the critical boundary into the American region, as the distance to the critical boundary becomes larger than $\sigma_1\sqrt{\epsilon}$ and $\sigma_2\sqrt{\epsilon}$. Thus, deep in the American region, we obtain

$$c(t_{i-1}, \mathbf{y}_{i-1}) - M(t_{i-1}, \mathbf{y}_{i-1}) = -\epsilon \left[\frac{\partial}{\partial t_i} + \hat{H}\left(t_{i-1}, \mathbf{y}_{i-1}\right)\right] c(t_i, \mathbf{y}_{i-1}). \tag{A21}$$

In the continuum time limit, $\epsilon \to 0$, the last equation becomes (A17) and is applicable to the whole American region when both $\sigma_1\sqrt{\epsilon} \to 0$ and $\sigma_2\sqrt{\epsilon} \to 0$.

Let us explicitly verify that the equation (A17) is valid right at the boundary on the American side in the continuum time limit. When the time is descritized, the presence of the lower cutoff for the time interval introduces two artificial length scales, $\sigma_1\sqrt{\epsilon}$ and $\sigma_2\sqrt{\epsilon}$. On these length scales, the discrete time model may contain features related to the particular discretizing scheme. These artificial features in the function $A(\tau, \mathbf{y})$ die out in the continuum time limit. Below we check the validity of the equation (A17) right at the boundary for one-factor models and point out the proof for multi-factor models.

Let us denote the critical boundary by $x^*(t)$, which forms a line in the continuous time limit for one-factor models. Within the European region, the Black-Scholes equation implies, for $\delta \sim \epsilon \ll 1$,

$$P_a(t-\epsilon, x+\delta) - P_a(t,x) = \left(\epsilon \hat{H}(t,x) + \delta \frac{\partial}{\partial x}\right) P_a(t,x) + \mathcal{O}(\epsilon^2), \tag{A22}$$

where $\mathcal{O}(\epsilon^2)$ indicates that the omitted terms are of order ϵ^2. Let us choose $\delta = x^*(t-\epsilon) - x^*(t)$ and approach the boundary from the American side,

$$\lim_{x \to x^*(t)} c(t-\epsilon, x+\delta) - c(t,x) = \left[\left(-\epsilon \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial x}\right) c(t,x)\right]_{x=x^*(t)}. \tag{A23}$$

The option cash flows should be continuous across the critical boundary. Let us adopt the convention that $x < x^*(t)$ is the European region. From equations (A22) and (A23), we find

$$\lim_{x \to x^*(t),\, x<x^*(t)} \left(\epsilon \hat{H}(t,x) + \delta \frac{\partial}{\partial x}\right) P_a(t,x) = \left(-\epsilon \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial x}\right) c(t, x^*(t)). \tag{A24}$$

If the critical boundary is a continuous and smooth curve, equation (A13) implies that the first order derivative of the option price with respect to x is continuous across the boundary,

$$\lim_{x \to x^*(t),\, x<x^*(t)} \frac{\partial}{\partial x} P_a(t,x) = \lim_{x \to x^*(t),\, x>x^*(t)} \frac{\partial}{\partial x} P_a(t,x) = \lim_{x \to x^*(t)} \frac{\partial}{\partial x} c(t,x). \tag{A25}$$

From the last two equations, we find

$$\lim_{x \to x^*(t),\, x<x^*(t)} \hat{H}(t,x) P_a(t,x) = -\frac{\partial}{\partial t} c(t, x^*(t)). \tag{A26}$$

In the continuous time limit, we expect the extension of the American option price $P_a(t,x)$ into the American region to satisfy

$$P_a(t,x) = c(t,x), \quad \text{for } x > x^*(t). \tag{A27}$$

In particular,

$$\frac{\partial^2}{\partial x^2}\left[P_a(t,x) - c(t,x)\right] = 0, \quad \text{for } x > x^*(t). \tag{A28}$$

From equations (A28) and (A26), we obtain

$$\frac{\sigma^2}{2}\left[\lim_{x\to x^*(t),\, x<x^*(t)} \frac{\partial^2}{\partial x^2} - \lim_{x\to x^*(t),\, x>x^*(t)} \frac{\partial^2}{\partial x^2}\right] P_a(t,x) = - \lim_{x\to x^*(t)} \left[\frac{\partial}{\partial t} + \hat{H}(t,x)\right] c(t,x). \tag{A29}$$

The derivatives of the European option price $P(t,x)$ are continuous across the boundary. The only contribution to the left side of the above equation comes from the early exercise premium term. It is straightforward to verify that

$$\frac{\sigma^2}{2}\left[\lim_{x\to x^*(t),\, x<x^*(t)} \frac{\partial^2}{\partial x^2} - \lim_{x\to x^*(t),\, x>x^*(t)} \frac{\partial^2}{\partial x^2}\right] \int_t^T d\tau \int_{x^*(\tau)}^{\infty} dy G(t,x|\tau,y) A(\tau,y) = A(t,x^*(t)). \tag{A30}$$

Thus, we obtain from equations (A29) and (A30),

$$A(t,x^*(t)) = - \lim_{x\to x^*(t)} \left[\frac{\partial}{\partial t} + \hat{H}(t,x)\right] c(t,x). \tag{A31}$$

For multi-factor models, the relation (A26) still holds true but with the corresponding multi-factor operator $\hat{H}(t,\mathbf{x})$, e.g. equation (37) for two-factor models. Similar to equation (A28), the extension of the pricing formula of the American option into the American region should match the exercise generated cash flows. The option value and the first order derivatives are still continuous across the boundary. The discontinuity for second and higher order derivatives across the boundary should be along the direction perpendicular to the boundary.

APPENDIX B: ANALYTIC EXPRESSIONS FOR BACKWARD INDUCTING CASH FLOWS OF AN EUROPEAN OPTION

In this section, we derive analytic expressions for backward inducting option cash flows over a finite time interval of length ϵ. To solve equation (64), we expand the solution as a power series in the small parameter $\sigma^2\epsilon$,

$$Q(t,x) = Q_0(t,x) + Q_1(t,x) + \cdots. \tag{B1}$$

The first two terms in the expansion satisfy the following partial differential equations,

$$-\frac{\partial}{\partial t}Q_0(t,x) + \left[m_0(t)x\frac{\partial}{\partial x} + r(t,x)\right]Q_0(t,x) = 0, \tag{B2}$$

$$-\frac{\partial}{\partial t}Q_1(t,x) + \left[m_0(t)x\frac{\partial}{\partial x} + r(t,x)\right]Q_1(t,x) = I(t,x), \tag{B3}$$

$$I(t,x) = \left\{\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - [m(t,x) - m_0(t)]\,x\frac{\partial}{\partial x}\right\}Q_0(t,x). \tag{B4}$$

In equation (B2) the function $m_0(t)$ is chosen as some measure of average mean reversion, which is independent of x. This enables us to obtain an analytic solution explicitly. We assume $|m(t,x) - m_0(t)| < \sigma^2$. In practice, we may choose to keep some simple dependence of x in the function m_0 while still maintain solubility.

If our goal is to derive the cash flows at time t_1 from the known function $Q(t_2,x)$ at time t_2, with $t_2 - t_1 = \epsilon$, the boundary conditions are

$$\lim_{t\to t_2,\, t<t_2} Q_0(t,x) = Q(t_2,x), \tag{B5}$$

$$\lim_{t\to t_2,\, t<t_2} Q_1(t,x) = 0. \tag{B6}$$

The solutions to equations (B2) and (B3) for the above boundary conditions are

$$Q_0(t_1,x) = Q\left(t_2, \frac{x}{\lambda_1(t_1,t_2)}\right)\exp\left[-\int_{t_1}^{t_2} d\tau\, r\left(\tau, \frac{x}{\lambda_1(t_1,\tau)}\right)\right], \tag{B7}$$

$$Q_1(t_1,x) = \int_{t_1}^{t_2} d\tau\, I\left(\tau, \frac{x}{\lambda_1(t_1,\tau)}\right)\exp\left[-\int_{t_1}^{\tau} d\tau'\, r\left(\tau', \frac{x}{\lambda_1(t_1,\tau')}\right)\right], \tag{B8}$$

where the time dependent function λ_1 is defined in equation (60).

For $R(t,x)$ and equation (65), the expansion in $\sigma^2\epsilon$ is

$$R(t,x) = R_0(t,x) + R_1(t,x) + \cdots. \tag{B9}$$

The differential equations are

$$-\frac{\partial}{\partial t}R_0(t,x) = \left\{[b(t)x_T^* - \widetilde{m}(t)x]\frac{\partial}{\partial x} - \widetilde{r}(t,x)\right\}R_0(t,x), \tag{B10}$$

$$-\frac{\partial}{\partial t}R_1(t,x) = \left\{[b(t)x_T^* - \widetilde{m}(t)x]\frac{\partial}{\partial x} - \widetilde{r}(t,x)\right\}R_1(t,x) + \widetilde{J}(t,x), \tag{B11}$$

where $b(t)$ and $\widetilde{r}(t,x)$ are defined in equations (66) and (68). In addition, we have introduced the following convenient notations,

$$\widetilde{m}(t) = m_0(t) + \frac{1}{(T-t)\lambda_2(t,T)}, \tag{B12}$$

$$\widetilde{J}(t,x) = J(t,x) + \left\{ \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - [m(t,x) - m_0(t)]\, x \frac{\partial}{\partial x} \right\} R_0(t,x), \tag{B13}$$

where $\lambda_2(t,T)$ and $J(t,x)$ are defined in equations (61) and (69). The definition of $\widetilde{m}(t)$ is similar to equation (67). We note that the equations (B2) and (B3) are formally special cases of (B10) and (B11). The boundary conditions are

$$\lim_{t\to t_2,\, t<t_2} R_0(t,x) = R(t_2,x), \tag{B14}$$

$$\lim_{t\to t_2,\, t<t_2} R_1(t,x) = 0. \tag{B15}$$

We recall that both $Q(t_2,x)$ and $R(t_2,x)$ are known components of the cash flow function at time t_2. We note that $R(T,x) = 0$ at the maturity $t_2 = T$. The solution for the time period (t_1,t_2) is

$$R_0(t_1,x) = R\left(t_2, \widetilde{x}(t_1,t_2,x)\right) \exp\left[-\int_{t_1}^{t_2} d\tau\, \widetilde{r}\left(\tau, \widetilde{x}(t_1,\tau,x)\right) \right], \tag{B16}$$

$$R_1(t_1,x) = \int_{t_1}^{t_2} d\tau\, \widetilde{J}\left(\tau, \widetilde{x}(t_1,\tau,x)\right) \exp\left[-\int_{t_1}^{\tau} d\tau'\, \widetilde{r}\left(\tau', \widetilde{x}(t_1,\tau',x)\right) \right], \tag{B17}$$

with

$$\widetilde{x}(t,t',x) = x_T^* \int_t^{t'} d\tau \frac{b(\tau)}{\widetilde{\lambda}_1(\tau,t')} + \frac{x}{\widetilde{\lambda}(t,t')}, \tag{B18}$$

$$\widetilde{\lambda}(t,t') = \exp\left[\int_t^{t'} d\tau\, \widetilde{m}(\tau) \right]. \tag{B19}$$

We note that the function $\widetilde{\lambda}(t,t')$ is defined in analogy to equation (60).

The boundary equation (47) can be solved analytically near the maturity to obtain the asymptotic form of the critical boundary. For one-factor models, we first calculate the cash flows from equations (B7), (B8), (B16), and (B17). Then we substitute the results into equation (63). The early exercise premium is calculated from equations (45) and (46). The algebra is tedious but straightforward. The result is presented in equations (48) and (49).

This is a remarkably simple result: The critical boundary for American options in one-factor models has a *universal* form. All model specifics and option specifics are captured in a single parameter τ_0.

APPENDIX C: CRITICAL BOUNDARY FOR AMERICAN PUT OPTION

Our goal is to obtain an analytic expression for the boundary function $g(u,v)$ for the American put option defined in equation (81). When either v or u is small, we can approximate the critical boundary with equation (90). Let us denote

$$f(u,g_0) = \sqrt{\frac{u}{8\pi}} - \frac{g_0(u)}{\sqrt{u}}, \tag{C1}$$

$$h(\lambda,u,g_0) = \sqrt{\frac{u\lambda}{8\pi}} - \frac{g_0(u)-g_0((1-\lambda)u)}{\sqrt{u\lambda}}, \tag{C2}$$

the equations to determine $g_0(u)$ and $g_1(u)$ are

$$\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{f^2(u,g_0)}{2}\right] + f(u,g_0)N\left(f(u,g_0)\right) = \sqrt{\frac{u}{8\pi}}\int_0^1 d\lambda N\left(h(\lambda,u,g_0)\right), \tag{C3}$$

$$N(f(u,g_0))\frac{g_1(u)}{\sqrt{u}} - \int_0^1 d\lambda \frac{g_1(u)-(1-\lambda)g_1((1-\lambda)u)}{4\pi\sqrt{\lambda}}\exp\left[-\frac{h^2(\lambda,u,g_0)}{2}\right] = K(u). \tag{C4}$$

The function $K(u)$ is independent of $g_1(u)$,

$$K(u) = \frac{\sqrt{u}}{2}\left[\frac{1}{8\pi}+\frac{g_0^2(u)}{u^2}\right]N(-f(u,g_0)) - \left[\frac{1}{8\pi}+\frac{g_0(u)}{u\sqrt{8\pi}}\right]\exp\left[-\frac{f^2(u,g_0)}{2}\right]$$
$$+\frac{1}{8\pi}\int_0^1 d\lambda\sqrt{\lambda}\left\{\exp\left[-\frac{h^2(\lambda,u,g_0)}{2}\right] - \sqrt{u\lambda}N(-h(\lambda,u,g_0))\right\}. \tag{C5}$$

Since both $g_0(u)$ and $g_1(u)$ are universal functions, we solve them numerically. Then we fit the numerical solution for $g_0(u)$ to the analytic expression (85). The resulting coefficients are

$a_1 = 0.0756616$	$a_2 = -1.3619$	$u_1 = 0.232$	$\text{error}_1 = 0.00093$
$b_0 = 0.160785$	$b_1 = -0.0457779$	$b_2 = 0.151697$	$b_3 = 0.192856$
$b_4 = 3.11388$	$b_5 = -2.07701$	$b_6 = 1.02961$	$b_7 = -0.483763$
$b_8 = 1.61957$	$b_9 = -1.33786$	$b_{10} = -0.17879$	$b_{11} = -1.25056$
$b_{12} = 1.69794$	$u_2 = 134$	$\text{error}_2 = 0.000043$	

The fitting error is defined as

$$\text{error} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{g_0(u_i) - g_i}{g_0(u_i)}\right)^2}, \tag{C6}$$

where g_i is the numerical soltuion at u_i and $g_0(u)$ is the analytical expression (85).

We fit the numerical solution for $g_1(u)$ to the following analytic expression

$$g_1(u) = c_0 + c_1 \ln(u) + \sum_{i=1}^{7} d_i \sqrt{u^i}. \tag{C7}$$

The resulting coefficients are

$c_0 = -0.309069$ $c_1 = 0.0114945$ $d_1 = 0.114417$ $d_2 = -0.0408211$

$d_3 = 0.0103069$ $d_4 = -0.00153471$ $d_5 = 0.000108396$ $d_6 = -2.27873 \cdot 10^{-7}$

$d_7 = -3.24225 \cdot 10^{-7}$

The fitting error is 0.0026, defined in the same way as (C6). The put option prices calculated from the boundary expressed in $g_0(u)$ and $g_1(u)$ are listed in Table III. The achieved accuracy should be sufficient for most practical applications. For the extremely demanding applications, one should use the boundary $g(u, v)$ given by equation (86). We have developed a computer program to calculate $g(u, v_i)$ for any given value v_i and then fit the numerical results to equation (85). We have obtained coefficients for the expression (85) for about 70 different v_i values as the basis of the polynomial interpolation. The option price calculated from interpolated boundary (86) has relative precision better than 10^{-5} for all cases.

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TABLES

TABLE I. Correspondence between interest rate model represented by equations (36) and (37) and quantum mechanics. Quantum mechanics in imaginary time is studied in quantum statistical mechanics.

Interest Rate Model	Quantum Mechanics in Imaginary Time
Interest rate driven by n random factors	Quantum particle in n space dimensions
Random variables	Space coordinates
Wiener process	Free particle
Volatility square	Particle mass
Short term interest rate	Scalar potential
Drift (a_1,a_2)	Vector potential
Option cash flow	Particle wave function
Cash flow (generalized Black-Schloes) equation	Schrödinger equation

TABLE II. Illustration of the accuracy and efficiency of calculating the price of American put options from the analytic expressions for the critical boundary. The current stock price and put strike price are both set to 100. The risk free rate is 5%. The results are shown for two options with different volatility σ and maturity T. The computing time is for a Pentium III 930MHz laptop.

	$\sigma = 60\%$	$T = 1$ year	$\sigma = 100\%$	$T = 5$ year
Calculation Method	Option price	Computing time (sec)	Option price	Computing time (sec)
Equations (86), (88)	21.1951	$1.4 \cdot 10^{-3}$	61.1675	$0.49 \cdot 10^{-3}$
CRR tree, 100 steps	21.2738	$1.4 \cdot 10^{-3}$	61.7312	$1.4 \cdot 10^{-3}$
CRR tree, 1000 steps	21.203	0.14	61.2244	0.14
CRR tree, 4000 steps	21.1971	2.3	61.1822	2.3
CRR tree, 16000 steps	21.1956	45	61.1716	45
CRR tree, 64000 steps	21.1952	960	61.1689	959

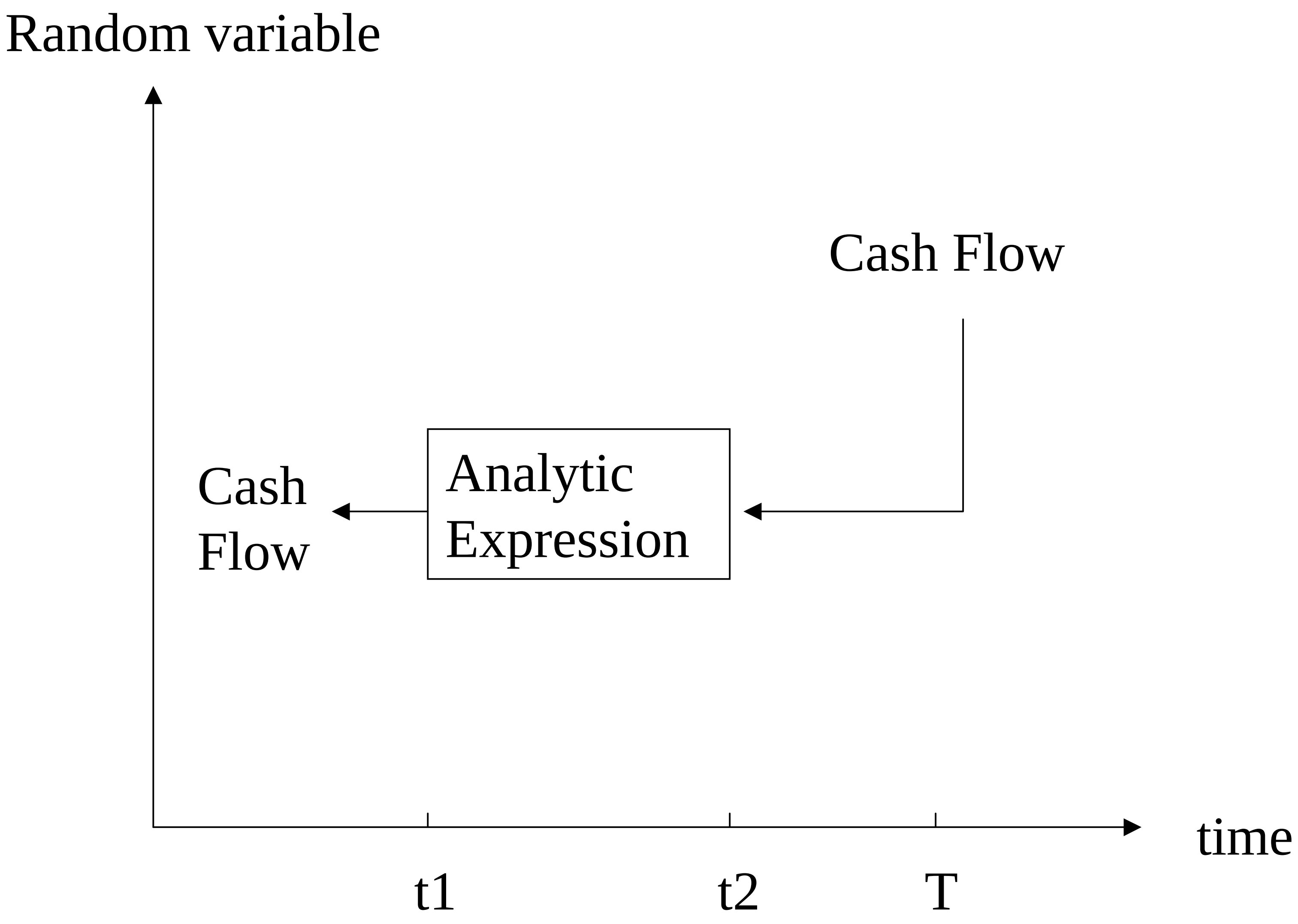
TABLE III. The price of American put options calculated from the critical boundary function $g(u,v) = g_0(u) + uvg_1(u)$ via equations (87) and (88), where $g_0(u)$ and $g_1(u)$ are two universal functions defined in Appendix C. The boundary of critical stock prices is expressed in terms of $g(u,v)$ through equations (73) and (81). The put options have maturity $T = 1$ or 2 years, risk free rate $r = 5\%$, and current stock price $S_0 = 100$. The put prices inside the bracket are calculated from the CRR binomial tree with 20000 time steps. For the extreme cases of high volatility and long maturity, such as 100% volatility and maturity of 5 years, the put option price should be calculated from the boundary function given by equation (86). Then the relative precision is better than 10^{-5} in all cases.

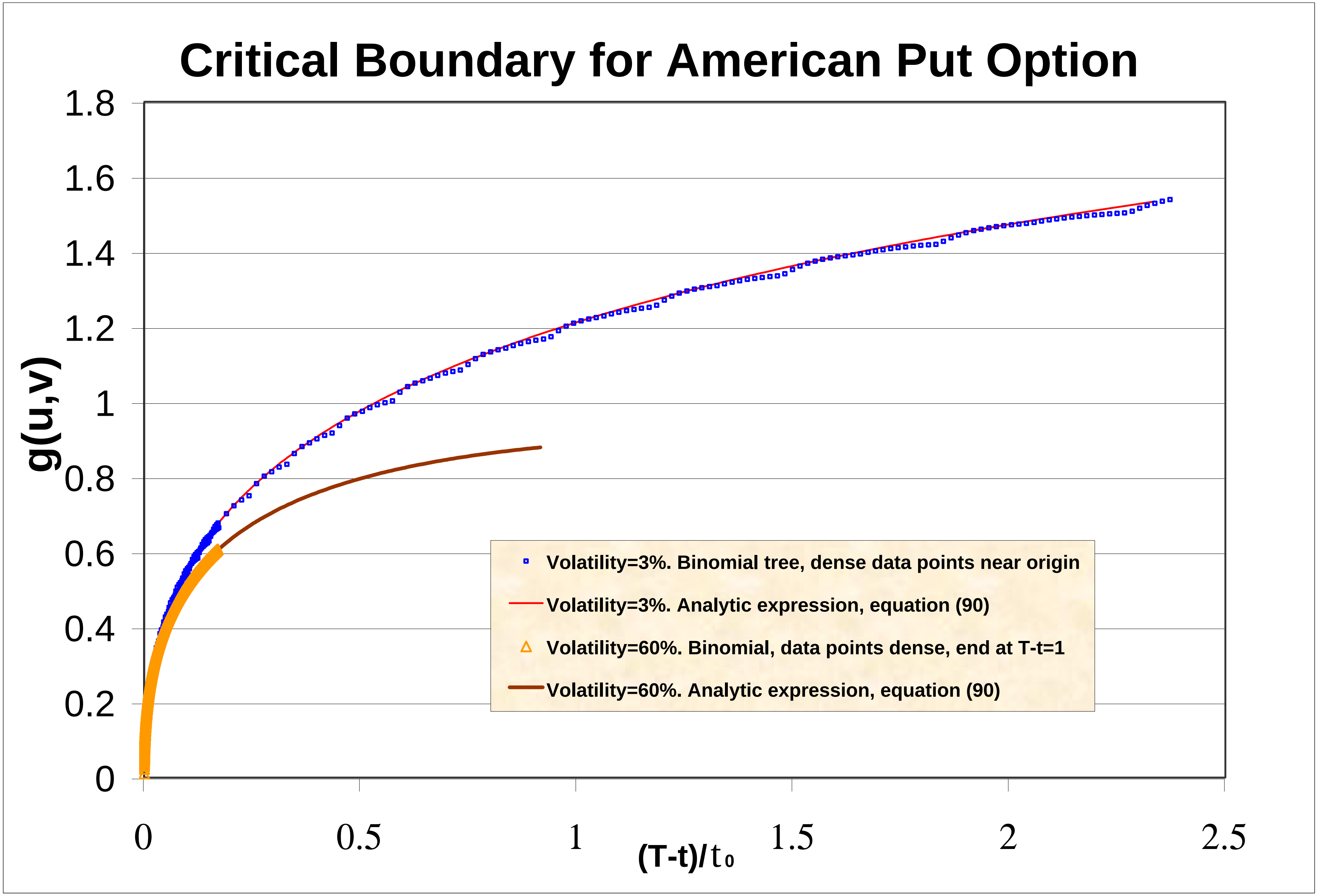
(σ,T)	Strike=50	Strike=80	Strike=100	Strike=120	Strike=150
$(3,1)$	0(0)	0(0)	0.32441(0.32443)	20.000(20.000)	50.000(50.000)
$(10,1)$	0(0)	0.00906(0.00906)	2.43681(2.43683)	20.000(20.000)	50.000(50.000)
$(30,1)$	0.04495(0.04494)	2.65706(2.65702)	9.87038(9.87023)	22.6827(22.6814)	50.0046(50.0000)
$(60,1)$	2.10962(2.10962)	11.0856(11.0855)	21.1959(21.1955)	33.9589(33.9571)	56.7850(56.7805)
$(80,1)$	5.06581(5.06584)	17.2095(17.2091)	28.5354(28.5344)	41.7745(41.7722)	64.2799(64.2745)
$(100,1)$	8.6263(8.62642)	23.2741(23.2734)	35.6060(35.6043)	49.3994(49.3960)	72.1094(72.1022)
$(80,2)$	10.2105(10.2097)	25.3591(25.3550)	37.7156(37.7071)	51.3761(51.3608)	73.7192(73.6910)

FIGURES

FIG. 1. Diagram illustrating the new computational approach developed in this paper for general Markovian interest rate models. The option time horizon is divided into a relatively small number of time intervals. Over a time interval (t_1, t_2), the analytic expressions for backward inducting option cash flows from time t_2 to time t_1 are derived. The calculation of the option price is reduced to numerically iterating the analytic expressions for one time interval.

FIG. 2. Comparison of the approximate expression (90) for the critical boundary, $g(u,v) = g_0(u) + uvg_1(u)$, to the numerical results calculated from the CRR binomial tree with 20000 time steps. We denote σ as the volatility and r as the risk free interest rate. The boundary only depends on two parameters: $\sigma^2(T-t)$ and $r(T-t)$, from which we form two new parameters $u = (T-t)/\tau_0$, $v^2 = \sigma^2\tau_0$. The characteristic time scale is $\tau_0 = \sigma^2/(8\pi r^2)$. The data points from the binomial tree for $\sigma = 60\%$ ends at $T - t = 1$ year. For $\sigma = 3\%$, the density of the data points from the binomial tree at $u > 0.18$ is reduced to enhance the clarity of the graph.


Random variable
Cash Flow
Cash
Flow
Analytic
Expression
time
t1
t2
T


Critical Boundary for American Put Option
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
g(u,v)
0
0.5
1
1.5
2
2.5
(T-t)/t0
Volatility=3%. Binomial tree, dense data points near origin
Volatility=3%. Analytic expression, equation (90)
Volatility=60%. Binomial, data points dense, end at T-t=1
Volatility=60%. Analytic expression, equation (90)