

# Bifurcation Routes in Financial Markets<sup>\*/</sup>

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## Abstract

The heterogeneity of expectations among traders introduces an important non-linearity into the financial markets. In a series of papers, Brock and Hommes, propose to model economic and financial markets as adaptive belief systems. Asset price fluctuations in adaptive belief systems are characterized by phases of close-to-the-fundamental-price fluctuations, phases of optimism where most agents follow an upward price trend, and phases of pessimism with small or large market crashes. In this paper will be discussed the EMH benchmark and forecasting rules of fundamentals and trend extrapolators. An illustrative example is supplied.

## Keywords

Heterogeneity of expectations, adaptive belief systems, forecasting rules, fundamentals, trend extrapolators equations, limit cycles, asymptotical stability

**JEL:** G10, G12, G14

## 1. Introduction

As a macroeconomic model describing conditions for equilibrium between the commodity market and the security market it is possible to use, among others, an IS-LM model. In a traditional Keynesian model, the commodity market is described by the so-called IS model (Investment-Saving Model), i.e. by means of the functions of savings  $S$  and investment  $I$ . The savings  $S(Y,R)$  are assumed to be an increasing function of the product  $Y$  and the nominal interest rate  $R$ . Similarly, the investment  $I(Y,R)$  is again an increasing function of the product  $Y$ , but a decreasing function of the nominal interest rate  $R$ . Considering a static IS model, we are looking for an equilibrium point, say  $(Y^*, R^*)$ , given by the following equation

$$S(Y,R)=I(Y,R), \quad (1)$$

i.e. for the equilibrium point we have  $S(Y^*, R^*)=I(Y^*, R^*)$ .

Similarly, the security market is described by the nominal supply of money  $M^s$  that must be equal to the nominal demand for money  $M^d$ . In a Keynesian model is assumed that the nominal money demand depends only on the product  $Y$  and the nominal interest rate  $R$ . The equilibrium in the money market is then given by

$$M^s = M^d(Y,R). \quad (2)$$

In dynamic settings, any dynamic economic system is possible to be considered as an expectation feedback system. A theory of expectation formation is therefore a crucial part of any economic model. The rational expectation hypothesis is the paradigm in the economic theory. The rational expectation hypothesis is very close related to the efficient market hypothesis. By the efficient market hypothesis there is no forecastable structure in asset returns because rational traders process information very quickly and this is reflected immediately in asset prices. Thus the value of a risky asset is completely determined by its fundamental price, equal to the present value of the expected future dividends. By this hypothesis all traders are rational and changes in asset prices are random. The hypothesis that all traders are rational is a little bit limiting. Hence a new alternative to the efficient market hypothesis, the heterogeneous market hypothesis (Hommes 2000), was introduced. Traders choose their trading strategy according to a measure expressing a coupling between market equilibrium dynamics and evolutionary updating of traders beliefs. For using formation of expectation in a security market we introduce the non-linear framework used by Chiarella (Chiarella(2000)) based on a portfolio adjustment imposed by non-linear asset demand functions. The security market model involving heterogeneous agents is used. We assume that on the security market exist two groups of agents:

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- a) fundamentalists ( $F$ ) who possess varying degrees of information about their economic background and using this in forming expectations,
- b) chartists ( $Ch$ ) who only use past price to form their expectations.

We firstly verify whether the adaptively evolving expectations of both fundamentalists and chartists in the heterogeneous market can lead to global stability. Using the approach of Brock and Hommes (Brock&Hommes (1997)) and we introduce

$$x_t = nF_t - nCh_t \quad (3)$$

with

$$x_t = \tanh\left(\frac{\delta}{2} \cdot \left[ (\pi_{t-\Delta,t} - E_F(\pi_{t-\Delta,t}))^2 - (\pi_{t-\Delta,t} - E_{Ch}(\pi_{t-\Delta,t}))^2 + cF - cCh \right]\right) \quad (4)$$

where

$$nF_t + nCh_t = \exp\left(\delta \cdot \left[ (\pi_{t-\Delta,t} - E_F(\pi_{t-\Delta,t}))^2 + cF \right]\right) + \exp\left(\delta \cdot \left[ (\pi_{t-\Delta,t} - E_{Ch}(\pi_{t-\Delta,t}))^2 + cCh \right]\right) \quad (5)$$

$$nF_t = \frac{nF_t}{nF_t + nCh_t}, \quad nCh_t = \frac{nCh_t}{nF_t + nCh_t}$$

the quantity  $\delta$  is the intensity of choice variable measuring how quickly agents switch between fundamentalism and chartism,  $cF$ ,  $cCh$  are costs incurred by each group in forming expectations. By some simple algebra we get

$$nF_t = \frac{1-x_t}{2}, \quad nCh_t = \frac{1+x_t}{2}. \quad (6)$$

The formation of expectations for fundamentalists is rational in the sense of knowledge of equilibrium. It means that fundamentalists form their estimation  $\tilde{\pi}_{t,t+\Delta}$  based on knowledge of the model. Thus

$$E_F \pi_{t,t+\Delta} = (1-\Delta \cdot \beta_{\pi_F}) \cdot (p^* - p_t) + \Delta \cdot \beta_{\pi_F} \cdot \tilde{\pi}_{t,t+\Delta}. \quad (7)$$

Fundamentalists anticipate that the further prices deviates from their fundamental value, the more likely they are to revert toward their fundamental value and the more likely to revert towards equilibrium. Thus

$$\Delta \cdot \beta_{\pi_F} = \exp\left(-\frac{(p^* - p_t)^2}{s}\right) \quad (8)$$

where  $s$  is the rate at which the fundamentalists switch between

$$\tilde{\pi}_{t,t+\Delta} \quad \text{and} \quad (p^* - p_t).$$

The classical adaptive scheme for chartists of the following form

$$E_{Ch} \pi_{t,t+\Delta} = (1-\Delta \cdot \beta_{\pi_{Ch}}) \cdot E_{Ch} \pi_{t-\Delta,t} + \Delta \cdot \beta_{\pi_{Ch}} \cdot \left(\frac{p_t - p_{t-\Delta}}{\Delta}\right) \quad (9)$$

is assumed. The value of the  $\Delta \beta_{\pi_{Ch}}$  belongs to the interval  $(0,1)$ , i.e.,

$$\Delta \beta_{\pi_{Ch}} \in (0,1) \quad (10)$$

We assume that  $\delta > 0$ ,  $cF > cCh$ , i.e., the fundamentalists incur a greater relative costs in forming their expectations.

In the next sections we present the complete macroeconomic model and the dynamics of this model in the section 2. A numerical example can be found in the section 3.

## 2. Dynamics in the complete macroeconomic model

Introducing a dynamic IS-LM model, we assume that the variables of the IS-LM model depend also on the current time  $t$ . Recall that in a neo-Keynesian approach it is assumed that the nominal interest  $R(t)$  depends also on the expected inflation  $\pi(t)$ , hence  $R(t)=r(t)+\pi(t)$  where  $r(t)$  is the real interest rate. The dynamic IS model is then given by the following differential equation

$$\frac{dY(t)}{dt} = \alpha \{I(Y(t), r(t)) - S(Y(t), r(t))\} \quad (11)$$

or by introducing the so-called propensity to invest  $i(\bullet, \bullet) = I(\bullet, \bullet)/Y(\bullet)$  and the propensity to save  $s(\bullet, \bullet) = S(\bullet, \bullet)/Y(\bullet)$ , also expressed by

$$\frac{dy(t)}{dt} = \alpha \{i(y(t), r(t)) - s(y(t), r(t))\} \quad (12)$$

where  $y(t) = \ln(Y(t))$  is called the output of the economy, and a positive constant  $\alpha$  signifies the speed of adjustment.

Observe that for an equilibrium point  $Y(t) = Y^*$ ,  $y(t) = y^*$ ,  $r(t) = r^*$  we have  $I(Y^*, r^*) = S(Y^*, r^*)$  or  $i(y^*, r^*) = s(y^*, r^*)$  (see (1)). We assume that

$$s(y, r) = s_0 + s_1 y + s_2 r \quad (13)$$

where the parameters  $s_i$ , ( $i=0,1,2$ , with  $s_1, s_2 > 0$ ) are known. It is convenient to assume that the propensity to invest  $i(y, r)$  is a product of  $1/(r+1)$  and so-called logistic function (Sladký&all(1999)). Hence the propensity to invest is assumed to be given analytically as

$$i(y, r) = \frac{1}{r+1} \cdot \frac{k}{1+b \cdot \exp(-ay)} \quad (14)$$

where the parameters  $k$ ,  $a > 0$  and  $b$  is an arbitrary real number.

As for the security market, we let  $m^d$  denote the demand for money and  $m^s$  be the supply of money. The market dynamics is then described here by the money supply and money demand of Cagan's model of monetary dynamics. The money supply is supposed to vary randomly according to

$$m_t^s = m_{t-\Delta}^s \cdot \exp(\sigma \cdot \sqrt{\Delta} \cdot \xi_t) \quad (15)$$

where  $\Delta$  is the length of the time interval,  $\sigma$  is the volatility of the money supply and  $\xi \sim N(0,1)$  is a random variable with a normal distribution,  $m_t = \ln(M_t)$ ,  $M_t$  is a nominal money at time  $t$ . The demand for money can be described either by the traditional Keynesian demand-for-money function in the following form

$$m_t^d = l(y_t, r_t) = l_0 + l_1 \cdot y_t - l_2 \cdot (r_t + \pi_t) \quad (16a)$$

or by Cagan's model as the money demand of time  $t$  being in the following form

$$m_t^d = p_t \cdot \exp(f(\pi_t)) \quad (16b)$$

where  $p_t = \ln P_t$  is an aggregate price level of time  $t$ ,  $\pi_t = E(dp_t/dt)$  and the function  $f$  is taken to be linear (Chiarella&Khamin(2000))

$$f(\pi_t) = c - d \cdot \pi_t \quad (17)$$

The price dynamics of the security market is governed by price adjustment to excess money demand as follows

$$dp(t)/dt = \beta \cdot (m^s - m^d) \quad (18)$$

The expected inflation  $\pi(t)$  is assumed to be adaptive as shown by the following differential equation

$$d\pi(t)/dt = \gamma \cdot (dp(t)/dt - \pi(t)) \quad (19)$$

where  $\gamma$  is a positive constant. Moreover, in a detailed analysis, as shown in Kodera (1996) or Sladký and all (1999), there exists a constant  $\mu$  such that

$$dp(t)/dt = \mu \cdot dy(t)/dt \quad (20)$$

From (19) we can conclude that

$$d\pi(t)/dt = \delta \cdot (\mu \cdot dy(t)/dt - \pi(t)). \quad (21)$$

Inserting from (13), (14) into (12) and from (16a) into (22b), we can conclude that the dynamics of the complete model is given by the following system of the four differential equations:

$$\frac{dy(t)}{dt} = \alpha \cdot \left\{ \frac{1}{r+1} \cdot \frac{k}{1+b \cdot \exp(-ay(t))} - [s_0 + s_1 y(t) + s_2 r(t)] \right\} \quad (22a)$$

$$\frac{dr(t)}{dt} = \beta \cdot \left\{ l_0 + l_1 \cdot y(t) - l_2 \cdot [r(t) + \pi(t)] - m_0^s \cdot \exp(\sigma \cdot \sqrt{\Delta} \cdot \sum_{l=0}^t \xi_{t-l\Delta}) \right\} \quad (22b)$$

$$\frac{d\pi(t)}{dt} = \delta \cdot \left\{ \mu \cdot \alpha \cdot \left[ \frac{1}{r+1} \cdot \frac{k}{1+b \cdot \exp(-ay(t))} - [s_0 + s_1 y(t) + s_2 r(t)] \right] - \pi(t) \right\} \quad (22c)$$

$$\frac{dp(t)}{dt} = \beta \cdot \left\{ m_0^s \cdot \exp(\sigma \cdot \sqrt{\Delta} \cdot \sum_{l=0}^t \xi_{t-l\Delta}) - p_t \cdot \exp(f(\pi_t) - \sigma \cdot \sqrt{\Delta} \xi_t) \right\} \quad (22d)$$

Now let us suppose that the system given by (22 a-d) is in equilibrium, i.e. there exist equilibrium points  $y^*$ ,  $r^*$ ,  $\pi^*$ , and  $p^*$  such that

$$\lim_{t \rightarrow \infty} y(t) = y^*, \lim_{t \rightarrow \infty} r(t) = r^*, \lim_{t \rightarrow \infty} \pi(t) = \pi^*, \lim_{t \rightarrow \infty} p(t) = p^* \quad (23)$$

To calculate the values  $y^*$ ,  $r^*$ ,  $\pi^*$ , and  $p^*$  the system of equations (22) take on the following form:

$$\frac{1}{r^*+1} \cdot \frac{k}{1+b \cdot \exp(-ay^*)} = [s_0 + s_1 y^* + s_2 r^*] \quad (24a)$$

$$l_0 + l_1 y^* + l_2 [r^* + \pi^*] = p^* \cdot \exp(c - d \cdot \pi^* - \sigma \cdot \sqrt{\Delta} \xi_t) \quad (24b)$$

$$\mu \cdot \alpha \cdot \left[ \frac{1}{r^*+1} \cdot \frac{k}{1+b \cdot \exp(-ay^*)} - [s_0 + s_1 y^* + s_2 r^*] \right] = \pi^* \quad (24c)$$

$$p^* = \frac{m_0^s \cdot \exp(\sigma \cdot \sqrt{\Delta} \cdot \sum_{l=0}^t \xi_l)}{\exp(f(\pi^*) - \sigma \cdot \sqrt{\Delta} \xi)} \quad (24d)$$

A solution of this system is a base for the fundamentalist forming expectation. Adaptively on this part is associated the part of the chartists. On the following example is shown that such system is global stable. We have to compute the Jacobian of the dynamical system (22a-d) at its equilibrium point and eigenvalues of this Jacobian. The Jacobian of the system (22) in equilibrium point has the following form (25).

$$\mathbf{A} = \begin{vmatrix} \frac{1}{r+1} \cdot \frac{k \cdot b \cdot \exp(-a \cdot y(t))}{(1+b \cdot \exp(-a \cdot y(t)))^2} - s_1 & \frac{1}{r+1} \cdot \frac{k}{1+b \cdot \exp(-a \cdot y(t))} + s_2 & 0 & 0 \\ l_1 & -l_2 & -1 & 0 \\ 0 & 0 & -1 & X \\ 0 & 0 & 0 & \sigma \cdot \sqrt{h} \cdot konst \end{vmatrix} \quad (25)$$

where

$$\begin{aligned}
X = & \left[ -\frac{1}{4} + \frac{1}{4} \tanh \left[ \frac{\delta}{2} \left[ \left( 9.8 \frac{m_s}{s^2} - \pi F \right)^2 - \left( 9.8 \frac{m_s}{s^2} - \pi C \right)^2 + cF - cC \right] \right]^2 \right] \cdot \\
& \cdot \frac{\delta \cdot 19.6}{s^2} (\pi C - \pi F) \cdot uC + \\
& + \left[ \frac{1}{4} - \frac{1}{4} \tanh \left[ \frac{\delta}{2} \left[ \left( 9.8 \frac{m_s}{s^2} - \pi F \right)^2 - \left( 9.8 \frac{m_s}{s^2} - \pi C \right)^2 + cF - cC \right] \right]^2 \right] \cdot \\
& \cdot \frac{\delta \cdot 19.6}{s^2} (\pi F + \pi C) \cdot uF
\end{aligned} \tag{26}$$

A stable or unstable behavior over time depends exclusively on roots of the characteristic equation. In our case the characteristic equation has the following form:

$$\lambda^4 - c_1 \cdot \lambda^3 + c_2 \cdot \lambda^2 - c_3 \cdot \lambda + c_4 = 0 \tag{27}$$

where

$$\begin{aligned}
c_1 = \sum_{i=1}^4 a_{ii} & \text{ elements of the matrix } \mathbf{A} & c_2 & \text{ sum of all second order principal minors of the matrix } \mathbf{A} \\
c_4 = \det(\mathbf{A}) & & c_3 & \text{ sum of all principal minors of the third order of the matrix } \mathbf{A}
\end{aligned}$$

Routh-Hurwitz method (Gandolfo (1997), p.221) gives necessary and sufficient stability conditions for this system, i.e., eigenvalues of this characteristic equation have to possess negative real parts. The characteristic equation possesses negative real parts if and only if the following expressions are held.

$$\Delta_1 = -c_1 > 0 \quad \Delta_2 = \begin{vmatrix} -c_1 & c_3 \\ 1 & c_2 \end{vmatrix} > 0 \quad \Delta_3 = \begin{vmatrix} -c_1 & -c_3 & 0 \\ 1 & c_2 & c_4 \\ 0 & -c_1 & -c_3 \end{vmatrix} > 0 \quad \Delta_4 = \begin{vmatrix} -c_1 & -c_3 & 0 & 0 \\ 1 & c_2 & c_4 & 0 \\ 0 & -c_1 & -c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{vmatrix} = c_4 \cdot \Delta_3 > 0$$

Numerical results are introduced in the following section.

### 3. Numerical example

Let in the expressions (22a), (22b), (22c), (22d),  $\alpha = 0.20$ ,  $\beta\beta = \gamma = 0.05$ ,  $\mu = 1-2$  and,  $k=0.4$  in (22a) and constant  $m$  in (22b), all the remaining parameters are given as follows:

$a = 0.5$ ,  $b = 1$ ,  $c = 0.4$ ,  $c_C = 0$ ,  $c_F = 0.5$ ,  $d = 1$ ,  $h = 0.008$ ,  $l_0 = 0.92$ ,  $l_1 = 0.65$ ,  $l_2 = 7.99$ ,  $\delta = 0.5$ ,  $\beta = 2.15$ ,  $\beta_C = 30$ ,  $\sigma = 0.01$ ,  $s = 0.99$ ,  $s_0 = -0.16$ ,  $s_1 = 0.1$ ,  $s_2 = 0.016$ ,  $\pi_C = 0.9$ ,  $\pi_F = 1.1$ .

Then the Jacobian matrix has the following form at equilibrium point ( $y^* = 2.04$ ,  $r^* = 0.093$ ,  $\pi^* = 1.55$ , and  $p^* = 2.48$ ):

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0.4 \cdot \exp(-0.5 \cdot 2.04) & -0.1 & \frac{1}{1.093} \cdot \frac{0.4}{1 + \exp(-0.5 \cdot 2.04)} + 0.016 & 0 & 0 \\ 1.093 & \frac{0.4 \cdot \exp(-0.5 \cdot 2.04)}{(1 + \exp(-0.5 \cdot 2.04))^2} & -0.1 & \frac{1}{1.093} \cdot \frac{0.4}{1 + \exp(-0.5 \cdot 2.04)} + 0.016 & 0 & 0 \\ & 0.65 & & -7.99 & -1 & 0 \\ & 0 & & 0 & -1 & 0.0019 \\ & 0 & & 0 & 0 & 0.01 \cdot \sqrt{0.008} \cdot konst \end{vmatrix} \tag{28}$$

A determinant of the matrix  $\mathbf{A}$  is equal 0.00018. By Routh-Hurwitz condition, the values of the  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$ ,  $\Delta_4$  are positive as is demonstrated on the following row

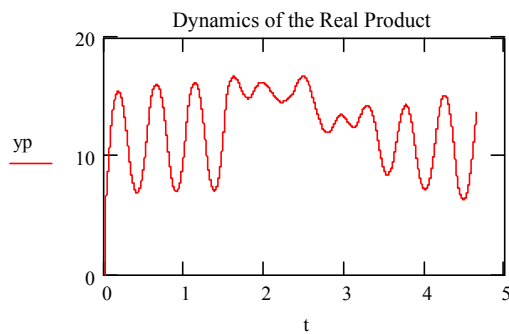
$$\Delta_1 = 4.041$$

$$\Delta_2 = 12.322$$

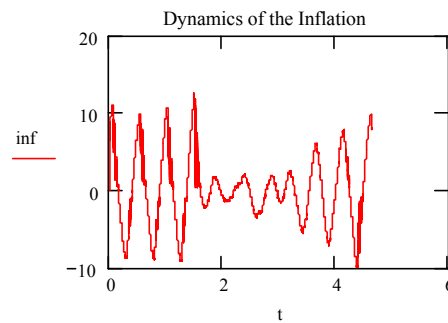
$$\Delta_3 = 0.136$$

$$\Delta_4 = 0.00018$$

therefore the eigenvalues are negative real parts. The eigenvalues are  $(-0.035, -8.007, -1, -0.00065)$ . Thus the system is a globally stable. It is possible to show that the stability condition is fulfilled for a wide region of the parameter space. In the figures 1 and 2 is shown the dynamics of real product and inflation. Here is demonstrated that at the beginning of the period was a percentage of the chartists greater than a percentage of the fundamentalists and the market system has had in this period the high amplitudes of cycles. An adjusting mechanism has acted towards increasing of the percentage of fundamentalists and this situation causes a stabilization activity in the economic system. The stable economic system with low amplitudes of cycles enables to make price – adaptive – pursuit strategy and thus a percentage of chartists in security market is increasing. Again a period of the high amplitudes of cycles is started.

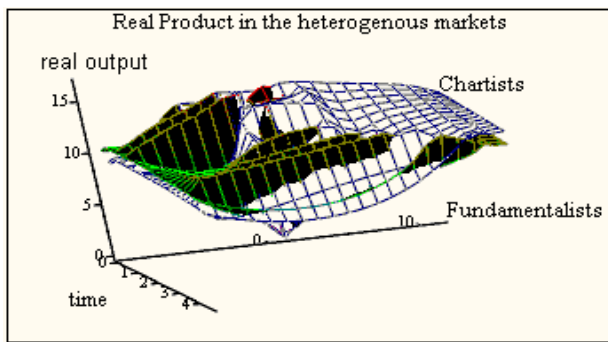


**Fig. 1**



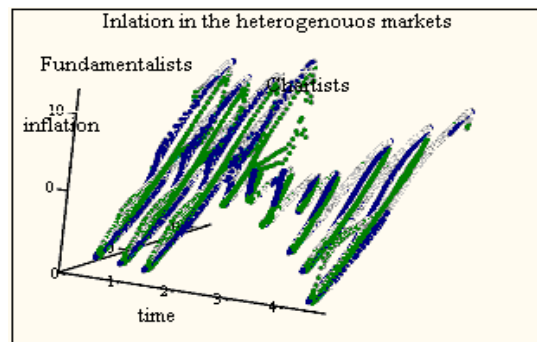
**Fig. 2**

In the figures 3, 4, and 5 this basic mechanism is observed when the fundamentalists put increasing weight on a reversion to the fundamental as the real product is very low, as inflation accelerates, and at the same time an increasing fraction of chartists is switching to fundamentalism. This mechanism performs stabilizing effects for this economic system. In the fig. 5 it is possible to observe the price-trend pursuit strategy of chartists whilst fundamentalists hold the fundamental-model position. There are higher amplitudes of cycle behavior of real product when chartists have a more weight on the heterogenous market.



$$(t, \pi_C, yp), (t, \pi_F, yp)$$

**Fig. 3**



$$(t, \pi_C, inf), (t, \pi_F, inf)$$

**Fig. 4**

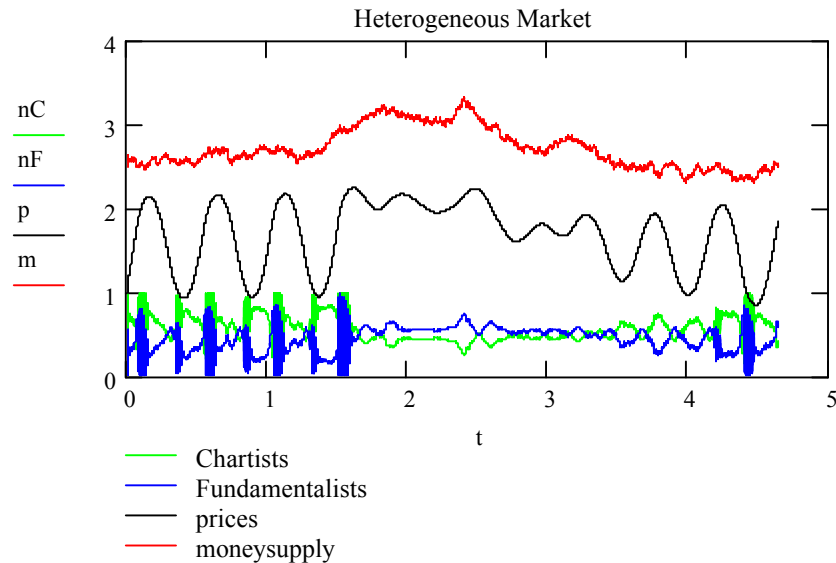


Fig. 5

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