The Relative Value Theory

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Abstract

I propose the following theory: rational investors allocate capital to maximize the probability-weighted geometric mean of payoffs. I present the theory's rationale and investigate its implications. I find that diversification increases cumulative returns and asset values are relative. The theory does not make assumptions about investors' risk preferences and explains the "risk premium" without using the utility function concept. In a world of rational, risk indifferent investors the market tends to a stable equilibrium characterized by all investors holding the market. I calculate equilibrium prices for a simple model. The theory uses a unique discount rate (to account for time value of money) and ignores the required rate of return concept. The theory addresses some of the CAPM's inconsistencies. I conclude the theory is coherent and useful in explaining observable finance phenomena.

Finance theory rests on the fundamental assumption that rational investors want to maximize the probability-weighted arithmetic mean of their portfolio's payoffs (arithmetic expected payoff).[1] Given a set of market prices, this rationality criterion is equivalent to maximizing the arithmetic mean of returns (arithmetic expected return). Empirical studies show that riskier assets have higher arithmetic expected returns.[2] To reconcile the rationality criterion with empirical results theorists introduce the utility function concept and assume investors are risk-averse.[3] The risk aversion assumption plays a key role in the CAPM theory.[4]

The proposed relative value theory (RVT) introduces the following rationality criterion: rational investors want to maximize the probability-weighted geometric mean of payoffs (geometric expected payoff). By payoff we mean the present value of an asset's stream of cash flows discounted at a unique rate representing time value of money. Given a set of market prices, RVT's rationality criterion is equivalent to maximizing the geometric mean of gross returns (geometric expected return).

For instance, the geometric expected return for a risky asset R with two possible gross returns, 1.3 with probability 0.6, and 0.65 with probability 0.4, is approximately 0.985, whereas its arithmetic expected return is 1.04 (all returns are gross returns):

$$GER = 1.3^{0.6} * 0.65^{0.4} \sim 0.985$$

$$AER = 0.6*1.3 + 0.4*0.65 = 1.04$$

According to current finance theory a rational, risk indifferent investor would prefer the risky asset R over a risk free R_f asset with a gross return of 1 (because R has a higher arithmetic expected return than R_f). According to the RVT this investor would instead prefer the risk free asset R_f (because R_f has a higher geometric mean than R).

By continually reinvesting the proceeds in the risky asset R an investor will end up loosing all his money. This result can be mathematically proved with limits or verified experimentally using a computer program. The example points to the geometric mean maximization being a superior criterion for rationality than the arithmetic mean maximization.

The case supporting the RVT appears to be particularly strong. It can be mathematically proved that maximizing the geometric expected return is the investing strategy that, over the long term, will consistently outperform any other strategy in terms of cumulative returns.

The RVT makes the utility function concept close to irrelevant. The RVT explains observable finance phenomena without referring to this concept and without assuming investors are risk-averse. The above-mentioned risky asset has a higher arithmetic expected return than the risk-free asset, but this risk premium is apparent since it cannot be used to outperform the risk free asset in terms of cumulative returns.

RVT's implications on finance theory are significant. Most notably, asset values are relative. The effect on cumulative returns of adding an asset to a portfolio depends on the correlation between the asset's cash flows and those of the rest of the portfolio. The RVT also indicates that diversification not only reduces risk but also increases cumulative returns, thus contradicting the view that returns can be improved by simply undertaking more risk.

The RVT provides the means to investigate the formation of asset prices in the market. Under the RVT assumption, it can be proved that, in a world of rational, risk indifferent investors, the market evolves toward a stable equilibrium characterized by all investors holding the market. Equilibrium prices can be determined without using the required rate of return concept. The equilibrium price formulas provide an explanation for the stock market consistently outperforming the risk free asset.

The RVT has a simple mathematical foundation, is applicable to any type of security and circumvents some of CAPM's weaknesses and inconsistencies.

The paper proceeds as follows. Section I presents RVT's rationale. Section II describes market equilibrium and price formation. Section III presents RVT's implications. Section IV runs a comparison between the CAPM and the RVT. Section V summarizes the paper's findings.

I. RVT's rationale

We consider a risky asset with two possible gross returns: R_+ with probability p_+, and R_- with probability p_- (p_- plus p_+ equals 1). We assume an investor can collect the proceeds and reinvest them in the same asset over and over again (we'll refer to a single step as an investing loop). After n investing loops each dollar initially invested we'll turn into R_n dollars (R_n is the gross return per dollar invested). It can be mathematically proved (please refer to Appendix A for a complete demonstration) that:

$$\lim_{n\to\infty} \sqrt[n]{R_n} = R_+^{\ p_+} \times R_-^{\ p_-} \tag{1}$$

In other words, the expected cumulative gross return per investing loop equals the geometric expected return. The higher the geometric expected return for a portfolio, the higher its expected cumulative return.

We consider two assets A and B. The geometric expected returns are GM_A and respectively GM_B, with $GM_A > GM_B$. We use R^A_n and R^B_n to indicate the gross cumulative returns after n investing loops. We use P^{AB}_n to indicate the probability of asset A outperforming asset B ($R^A_n > R^B_n$) after n investing loops. We can now use identity (1) to prove (please see Appendix B for a complete demonstration) that:

$$\lim_{n\to\infty} P_n^{AB} = 1 \tag{2}$$

In other words, the probability of portfolio A outperforming portfolio B after n investing loops tends to 1 when the number of loops tends to infinite. We can conclude that maximizing the geometric expected return is the investing strategy that will consistently outperform any other strategies over the long term. Since individual investors appear to be interested in investing strategies that consistently work, the above result provides credibility to RVT's rationality criterion.

The arithmetic expected return has a different meaning. We consider m investors that invest in identical assets (in terms of probability distribution) with uncorrelated payoffs. When m tends to infinite, the gross return for the investing community as a whole (total final wealth over total initial wealth) after one investing loop tends to the arithmetic expected return of the considered class of assets. If all investors have the same initial capital C, we can write (the demonstration is similar to the one provided in Appendix A for the geometric mean):

$$\lim_{n\to\infty} \frac{R_m}{m \times C} = p_+ \times R_+ + p_- \times R_- \qquad (3)$$

In other words, the higher the arithmetic expected return the higher the gross return for the investing community as a whole.

We conclude that maximizing the geometric expected return benefits the individual investor, whereas maximizing the arithmetic expected return benefits the investing community as a whole. It is reasonable to assume that individual investors are primarily concerned with their own financial situation, the financial situation of others being a secondary concern. In light of the above, RVT's rationality criterion appears reasonable.

The RVT's validity is further supported by its ability to explain the observed "risk premium" without referring to investors' utility functions or postulating investors' aversion to risk. As we previously mentioned, assets with higher arithmetic expected returns than the risk free asset do not necessarily outperform the risk free asset in terms of cumulative return. Consequently the observed "risk premium" (or at least most of it) is only apparent. To illustrate the above result we consider a risky asset that returns either 1.375 with probability 0.5 or 0.88 with probability 0.5. The arithmetic expected return is 1.1275, and according to the CAPM theory an investor must be risk-averse to prefer a risk free asset that returns only 1.1. However, the geometric expected return of the risky asset is also 1.1. Consequently, when the number of investing loops tends to infinite, the risky asset will perform in line with the risk free asset. Using (2) we observe that the risky asset will outperform any risk free asset with a gross return lower than 1.1 and will under perform any risk free asset with a gross return higher than 1.1. Since a higher arithmetic expected return does not guarantee a better performance in terms of cumulative returns we conclude the observed risk premium is only apparent.

The main weakness of the RVT is its reliance on the assumption that the market as a whole cannot go to zero. In other words, there must be a risk free asset embedded in the market for the RVT to function. Should this fundamental assumption be violated, the geometric mean for the market as a whole would be zero. Moreover, the geometric mean would be zero for all portfolios since no portfolio can be construed to avoid entirely the possibility of a zero payoff. We believe the weakness is minor. We could defend the RVT by noting the market going to zero would imply the ceasing of economic activity, which in turn would imply the end of mankind. In such an extreme situation investor's initial capital allocation would not matter anyway. We conclude that investors can ignore this possibility altogether.

The RVT's validity is also supported by the simplicity of its mathematical foundation, and, as later discussed in the paper (section II), by its ability to explain why the market provides a higher cumulative return than the risk free asset. The RVT also circumvents some of the weaknesses and apparent inconsistencies of the CAPM theory.

II. A simple world with two risky assets

To illustrate how prices are formed and equilibrium is reached, we consider a simple world with two risky assets A and B. The future evolution of this a world can follow n different scenarios (states). In scenario i (i goes from 1 to n) the payoffs of assets A and B are A_i and respectively B_i

(where A_i, and B_i are all positive numbers). By payoff we mean the present value of the stream of cash flows discounted at a unique rate T representing time value of money.

We note p_i the probability of state i (the n probabilities p_i add up to 1). Investors can freely trade ownership in these assets in the market. We further assume that A_i and B_i cannot all be zero at the same time (a state in which A_i plus B_i equals 0 does not exist).

We define equilibrium as a situation in which no transaction exists that will increase the geometric expected payoff for both investors engaged in it. In other words equilibrium is reached when all transactions will decrease the geometric expected payoff for at least one of the two investors.

We consider two rational investors (as per RVT's definition). Investor 1 owns X_A of asset A, and X_B of asset B; investor 2 owns Y_A of asset A, and Y_B of asset B. We say that investors 1 and 2 have the same capital allocation if:

$$X_A / Y_A = X_B / Y_B = p, \text{ where } p > 0 \tag{4}$$

We prove that unless the two investors have the same capital allocation as defined by (4) they will be able to transact to increase the geometric expected payoff for both their portfolios. Consequently individual investors will keep transacting until the differences in capital allocation will disappear and all investors will hold the same portfolio (please refer to Appendix C for a full demonstration of the statements in this paragraph).

Since the cumulative investor ownership in any given asset must add up to one (representing 100% ownership) it is obvious that, at equilibrium, each individual investor will hold the market. An investor owning the market means his percentage ownership will be the same for any given asset. For investors 1 and 2 mentioned above we write:

$$X_A = X_B = X, \text{ and } Y_A = Y_B = Y, \text{ where } X / Y = p \tag{5}$$

So far we showed that, under the RVT assumption, the market would necessarily evolve toward a stable equilibrium in which all investors own the market. Considering the above results we now want to determine asset prices at equilibrium.

At equilibrium individual investors can not increase their portfolio's geometric expected payoff by trading assets at market prices. By setting this condition for every possible transaction (every possible pair of assets), and by setting the total value of the market as a whole (market value of asset A plus market value of asset B) equal to its geometric expected payoff[5], we obtain a system of two linear equations with two unknowns. The two unknowns are the equilibrium market prices of the assets A and B. The solution to this system exists, and is unique.

We consider the general case of a world with m risky assets A^j (j goes from 1 to m), and n possible states i (i goes from 1 to n) with probabilities p_i. Time value of money is unique and measured by the discount rate T. We use A^j_i to denote the present value of asset A^j's stream of cash flows in state i. Under the RVT assumption the equilibrium market price of asset A^j is given by (please refer to Appendix D for the related mathematics):

$$P_{A^j} = \left(\sum_{i=1}^{n} p_i \frac{A_i^j}{M_i} \right) \cdot GM \tag{6}$$

where

$$M_i = \sum_{k=1}^{m} A_i^k \text{ (the total payoff of the market in state i),}$$

and

$$GM = \prod_{i-1}^{n} M_i \text{ (the geometric mean of payoffs for the market as a whole).}$$

Consequently the geometric expected return of asset A^j is given by (after simplifications):

$$GR_{A^j} = \frac{GM(P_i)}{AM(P_i)} \tag{7}$$

where

$$P_i = \frac{A_i}{M_i},$$

$$GM(P_i) = \prod_{i=1}^{n} P_i^{\,p_i} ,$$

and

$$AM(P_i) = \sum_{i=1}^{n} p_i P_i \ .$$

Summarizing, we proved that, under the RVT assumption, the market evolves toward a stable equilibrium characterized by all investors holding the market and asset prices given by (6).

We observe that the geometric expected return of any asset or portfolio is lower than the geometric expected return of the market as a whole. Since the geometric mean is always smaller than the arithmetic mean, equation (7) indicates that the geometric expected return of any given asset is smaller than one, whereas the geometric expected return of the market is equal to one.

Consequently considering (2) the market as a whole will outperform any given asset (including the risk free asset) in terms of cumulative returns. The RVT therefore provides a viable explanation for the market outperforming the risk free asset over the past century.

We must note that the market value of any risky asset depends on the existence and nature (most importantly payoff correlation) of all other risky assets. Adding a new risky asset to the market will increase the market value of the existing assets. This effect also contributes to the market outperforming the risk free asset in periods of economic development in which new companies and new industries are born. Obviously the past century represents such a period. This result provides further support to the validity of the RVT.

III. Implications on finance theory

The RVT has significant implications, some of them being discussed below.

A. Diversification not only decreases risk but also increases cumulative returns

According to the CAPM theory diversification brings benefits by reducing risk. Lower risk is considered a benefit because of CAPM's basic assumption that investors are risk-averse. According to the RVT, diversification not only reduces risk but also increases expected cumulative returns. Consequently rational, risk-indifferent (even risk-loving) investors will still diversify their holdings. The following example illustrates this effect.

We consider two investments A and B. Investment A returns either 1.4 with probability 0.5 or 0.8 with probability 0.5. Investment B returns either 1.35 with probability 0.5 or 0.79 with probability 0.5. The outcomes of the two investments are not correlated. Although A is a clearly better investment than B, the best portfolio according to the RVT is some combination of the two. Indeed we can verify that allocating 90% of capital to investment A and 10% to investment B will result in a higher geometric expected return than allocating all capital to investment A. The result is somewhat counterintuitive and reveals the descriptive power of the RVT.

B. There is no absolute value for assets

Assets do not have an absolute value. Knowing the possible streams of cash flow and related probabilities is not sufficient to compute a price that all investors will agree upon. The equilibrium prices formula presented in section II clearly indicates a dependence on the nature of the other existing assets. An investor will be willing to pay different prices for the same given asset depending on the correlation between the cash flows of the asset and those of the investor's portfolio. The following example illustrates the relativity of asset values.

We consider an envelope that contains either 200$ with probability 0.5 or 50$ with probability 0.5. We consider an investor with a total capital of 100$ in cash. For simplicity we assume that the gross return of keeping the capital in cash is 1 with probability 1 (no return). The geometric expected return for a portfolio consisting of 100$ in cash is 1. This investor will be willing to pay 40$ for a 40% ownership in the envelope because the transaction will increase the geometric expected return of his portfolio. The gross return of the resulting portfolio (60$ in cash plus 40% ownership of the envelope) will be either 1.4 with probability 0.5 or 0.8 with probability 0.5. The geometric expected return will be:

$$GER = 1.4^{0.5} * 0.8^{0.5} \sim 1.0583 > 1$$

The same investor will not be willing to pay another 40$ for an additional 40% ownership in the envelope because the transaction will decrease the geometric expected return of his portfolio. The gross returns of the resulting portfolio (20$ in cash plus 80% ownership in the envelope) will be 1.8 with probability 0.5 and 0.6 with probability 0.5. The geometric expected return will be:

$$GER = 1.8^{0.5} * 0.6^{0.5} \sim 1.0392 < 1.0583$$

The above example shows that an individual investor will value an asset (in our case 40% ownership in the envelope) differently depending on the asset's correlation with the rest of his portfolio. Hence the value of an asset is relative.

C. The required rate of return is an artificial concept

The RVT does not make reference to the required rate of return concept. The RVT refers only to time value of money, which is unique and allows the present value calculation for any cash flow in the future, risky or not. Intuitively a rational, risk indifferent investor will not require a rate of return but will simply try to maximize his cumulative return.

We revisit the case of the envelope containing either 200$ with probability 0.5 or 50$ with probability 0.5. We assume the envelope trades in the market long before the date when the owner is allowed to open it and cash its content. The CAPM theory cannot be used to value such a security.

Just before its opening the envelope cannot be valued at more than 200$ or less than 50$. No matter how the CAPM values the envelope just before its opening, discounting its value at a rate higher than the risk free rate would result in an arbitrage opportunity. At an early enough point in time the envelope would trade for less than the value of the 50$ risk free asset embedded in the envelope. The CAPM appears inferior to the RVT for this particular type of asset.

IV. Weaknesses of the CAPM

The CAPM theory was thoroughly studied and is broadly accepted by academia and the investing community in general. This popularity can make the CAPM theory a self-fulfilling prophecy. At this point any empirical test would do nothing but confirm the CAPM theory. Such a confirmation would be a weak proof for the CAPM being the best approach to asset valuation.

The CAPM theory appears to be weak. The following paragraphs describe some arguments against the validity of the CAPM theory. None of these shortcomings apply to the RVT.

A. The CAPM theory appears to be inconsistent with efficient markets

It is reasonable to assume that some investors are risk indifferent and some are risk loving. In a highly efficient market the cross-section of risk loving investors should be large enough to create an efficient market for any risky stock. The risk loving investors will always outbid the risk-averse. As a consequence, the cash flows associated with riskier stocks will be discounted using a lower rate of return (presumably lower than the risk free rate). This fact would contradict the CAPM theory.

B. Risk indifference should be the equilibrium stance

Intuitively, being risk loving or risk-averse should come at a cost. A risk loving person will overpay for risky stocks thus accepting lower rates of return. A risk-averse person will forego good investment opportunities because of the associated risk. On the other hand, a risk indifferent person will be free to focus solely on maximizing returns. This constraint-free approach should lead to a better performance, which in turn should increase risk indifferent investors' wealth and consequently influence on the market as a whole. This force should drive the market away from any risk loving or risk-averse stance to an equilibrium characterized by risk indifference.

C. Companies cannot be valued without a market in place

The CAPM theory does not allow investors to value a business in the absence of the equity market because without a market in place there would be no beta coefficient and no market premium. As a consequence investors would not know how much to pay, or how much to ask, for any given business. The CAPM theory seems to suggest that the market is endowed with a knowledge and reasoning of its own, apart from the cumulative knowledge and intelligence of all investors. This fact runs against common sense.

D. Beta is not a good indicator of risk

The beta coefficient incorporates the effect on price of new information. Bad news can drive the price of a stock down on a day when the market is up, thus polluting the very correlation beta tries to detect.

Beta is calculated using daily movements in stock price and market index. If beta is calculated using two-day periods (or half-day periods) the resulting value might be totally different from the "real" beta. This fact questions beta's meaning and usefulness.

By looking at beta, investors "look in the rear mirror" instead of focusing on the future. If the market has no memory, beta should be useless in estimating risk.

E. The CAPM theory critically depends on the assumption investors are risk-averse

The CAPM theory cannot function in a risk indifferent market. However a risk indifferent world is at least theoretically possible. The CAPM theory is unable to provide a general framework for asset valuation, independent of risk preference assumptions.

V. Conclusions

Unlike the CAPM theory, which assumes rational investors want to maximize the probability-weighted arithmetic mean of their portfolio's payoffs, the RVT assumes rational investors want to maximize the probability-weighted geometric mean of their portfolio's payoffs.

We find this theory reasonable. The paper provides a mathematical demonstration of the fact that maximizing the geometric expected return is equivalent to maximizing long term cumulative returns. We find it reasonable to assume investors are primarily interested in their own long-term cumulative returns.

The RVT provides a simple explanation for the "risk premium" without referring to investors' utility functions or risk preferences. Basically the RVT shows that the risk premium (higher arithmetic expected returns for riskier assets) is apparent in the sense it does not allow investors to outperform the market over the long term (on a cumulative return basis).

The RVT does not use the elusive required rate of return concept, resting instead on a simple mathematical foundation that uses only two basic concepts: time value of money, and the probability distribution of payoffs. This simple mathematical foundation allows the theoretical investigation of how prices are formed and market equilibrium is reached.

We proved that in a world of rational, risk indifferent investors the market tends toward a stable equilibrium characterized by all investors owning the market. The RVT also shows that asset values are relative, and that diversification not only reduces risk but also increases cumulative returns. The RVT is applicable to any type of financial security.

The RVT circumvents some of CAPM's weaknesses and inconsistencies as described in section IV of the paper. RVT's main weakness consists in its critical reliance on the assumption the market cannot go to zero.

The RVT does not try to predict asset prices' evolution in the market. According to the RVT, at any given point in time market prices are determined by the perceived probability distribution of asset payoffs. Obviously the evolution of these probability distributions cannot be predicted and consequently neither can market prices.

We conclude that the RVT rests on a solid logical foundation, is consistent with most finance phenomena, and provides a workable theory of investing.

Appendix A

In this appendix, we prove the cumulative return per investing loop tends to the probability weighted geometric mean of returns when the number of loops tends to infinite:

$$\lim_{n\to\infty} \sqrt[n]{R_n} = R_+^{\,p_+} \times R_-^{\,p_-} \tag{A1}$$

After n investing loops the asset will return R_+ a total of n_+ times and will return R_- a total of n_- times. We have that n_+ plus n_- is equal to n. We have that:

$$R_n = R_+^{\,n_+} R_-^{\,n_-} \tag{A2}$$

Replacing R_n from (A2) in (A1) the limit becomes:

$$\lim_{n\to\infty} \sqrt[n]{R_n} = R_+^{\left(\lim n_+/n\right)} R_-^{\left(\lim n_-/n\right)} \tag{A3}$$

We have that:

$$\lim_{n\to\infty} {n_+}/{n} = p_+ \text{ and } \lim_{n\to\infty} {n_-}/{n} = p_- \qquad \text{(A4)}$$

Replacing (A4) in (A3) we conclude that:

$$\lim_{n\to\infty} \sqrt[n]{R_n} = R_+^{\ p_+} R_-^{\ p_-} \qquad \text{Q.E.D.}$$

We must note that this demonstration can be easily generalized to assets with many possible returns.

Appendix B

In this appendix, we prove that if GM_A is greater than GM_B then the probability P^{AB}_n of asset A outperforming asset B after n investing loops tends to 1 when the number of loops tends to infinite. Since GM_A is greater than GM_B a positive number R exists so that:

$$GM_A > R > GM_B \qquad \text{(B1)}$$

We use R_f to indicate a risk free asset with a gross return equal to R. Since R is risk free its gross return after n investing loops is R^n (R to the power of n).

Considering that GM_A is greater than R, a small positive number ε exists so that:

$$GM_A - \varepsilon > R \qquad \text{(B2)}$$

We know that:

$$\lim_{n\to\infty} \sqrt[n]{R_n^A} = GM_A \qquad \text{(B3)}$$

Using the definition of limits we can say that no matter how small a positive number ε, a positive integer n_ε exists so that for every integer n greater than n_ε we have that:

$$\sqrt[n]{R_n^A} > GM_A - \varepsilon \qquad \text{(B4)}$$

Given (B2), (B3) and (B4) we conclude that a positive integer n_ε exists so that for every integer n greater than n_ε we have that:

$$R_n^A > R^n \qquad \text{(B5)}$$

We can similarly show that another positive integer n'_ε exists so that for every integer n greater than n'_ε we have that:

$$R^n > R_n^B \qquad \text{(B6)}$$

Using (B5) and (B6) we can now conclude that for every integer n greater than Max (n'_ε , n'_ε) we have that R^A_n is greater than R^B_n, which is equivalent with:

$$\lim_{n\to\infty} P_n^{AB} = 1 \qquad \text{Q.E.D.}$$

Appendix C

In this appendix, we prove that unless two investors have the same capital allocation they will be able to transact to increase the geometric mean of payoffs for both their portfolios. Consequently, they will end up owning the same portfolio (same capital allocation).

We use O_A, and O_B for the cumulative ownership of the two investors in each of the two assets. We have that:

$$x_A + y_A = O_A,\ x_B + y_B = O_B \tag{C1}$$

The two investors can transact freely and consequently divide the cumulative portfolio into two portfolios. Considering (C1) the pair x_A and x_B describes all possible ways of splitting the cumulative portfolio. We use F as a notation for the sum of the geometric means of the two individual portfolios. Using (C1) we can write F as a function of x_A and x_B:

$$F(x_A, x_B) = \prod_{i=1}^{n}(A_i x_A + B_i x_B)^{P_i} + \prod_{i=1}^{n}[A_i O_A + B_i O_B - (A_i x_A + B_i x_B)]^{P_i} \tag{C2}$$

We operate the following substitutions:

$$A_i = A_i O_A,\ B_i = B_i O_B,\ x_A = x_A / O_A,\ x_B = x_B / O_B \tag{C3}$$

We rewrite (C2) as follows:

$$F(x_A, x_B) = \prod_{i=1}^{n}(A_i x_A + B_i x_B)^{P_i} + \prod_{i=1}^{n}[A_i + B_i - (A_i x_A + B_i x_B)]^{P_i} \tag{C4}$$

where $x_A,\ x_B \in [0,1]$.

For all the points in the definition space of function F that satisfy the condition x_A equals x_B (the points along the bisector) the value of F is constant and equal to the geometric expected payoff of the cumulative portfolio of the two investors:

$$F(m,m) = \prod_{i=1}^{n} (A_i + B_i)^{P_i},\ \text{for all}\ m \in [0,1] \qquad \text{(C5)}$$

We will prove that function F has a unique (and consequently global) local maximum along the bisector. Before proceeding with the demonstration it is useful to present a 3D graph of function F for a simple case with two assets and three possible states. The probability distribution of payoffs for this simple case is included in Table 1.

Table I Probability distribution of payoffs

The table describes a simple world with two risky assets and three possible states. Columns A and B indicate an asset's payoffs in each particular state. Rows 1, 2, and 3 indicate all the asset payoffs in a given state. The last column indicates the probabilities for each state.

Assets / States	A	B	Probabilities
1	$100	$10	0.1
2	$120	$80	0.5
3	$0	$300	0.4

A 3D graph of function F is provided in Figure 1. The graph of function F is shaped like a wave oriented along the bisector of the definition space. Changing the probability distributions of payoffs does not affect the shape of the graph in a fundamental way (even when considering extreme situations).

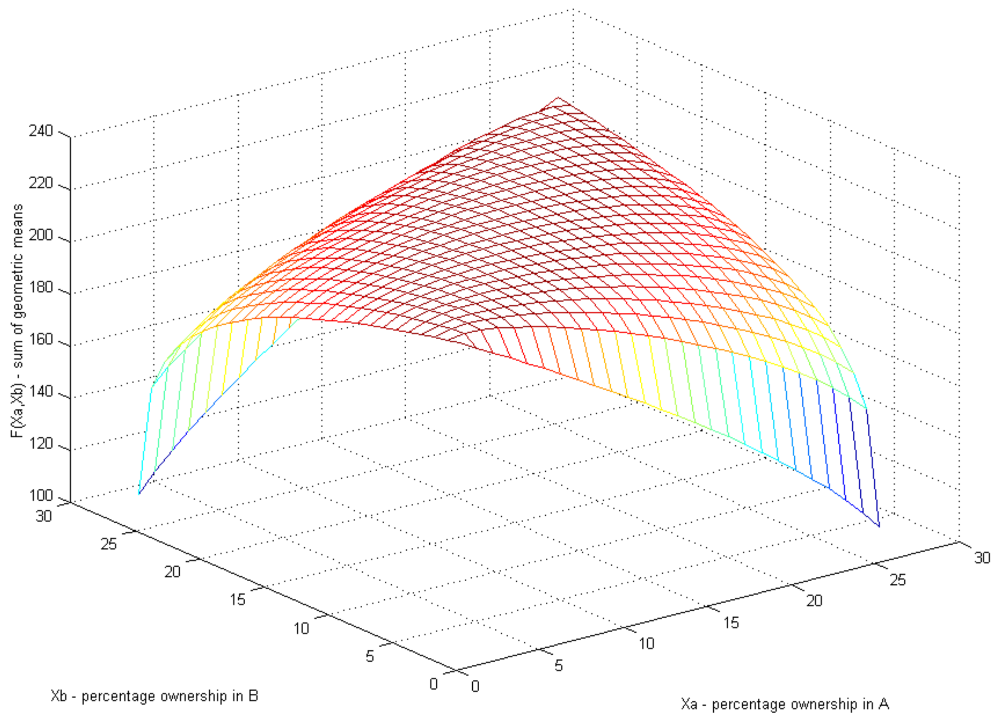

Figure 1 3D graph of function F

The figure illustrates the sum of the geometric expected payoff for the two investors as a function of their capital allocation. Xa and Xb are the percentage ownership of investor 1 in asset A and respectively asset B. Since investor 2 owns the remaining portion of the two assets, the pair Xa and Xb entirely describes the capital allocation of the two investors. We notice that the sum of the geometric means is maximized when the two investors own the market (which is along the bisector).

We can simplify formulas by using the following notations:

$$X_i = A_i x_A + B_i x_B,\ T_i = A_i + B_i,\ \overline{X} = \prod_{i=1}^{n} X_i^{p_i},\ \text{and}\ \overline{T - X} = \prod_{i=1}^{n} (T_i - X_i)^{p_i} \qquad \text{(C6)}$$

Considering (C6) we can now write F as:

$$F(x_A, x_B) = \overline{X} + \overline{T - X} \qquad \text{(C7)}$$

We use F^I_A and F^{II}_A for the first and respectively second degree derivatives of F in x_A. We can write:

$$F_A^I(x_A, x_B) = \sum_{i=1}^{n} p_i A_i \left[\frac{\overline{X}}{X_i} - \frac{\overline{T - X}}{T_i - X_i} \right] \qquad \text{(C8)}$$

$$F_A^{II}(x_A, x_B) = \overline{X} \left[\left(\sum_{i=1}^{n} \frac{p_i A_i}{X_i} \right)^2 - \sum_{i=1}^{n} p_i \left(\frac{A_i}{X_i} \right)^2 \right] + \overline{T - X} \left[\left(\sum_{i=1}^{n} \frac{p_i A_i}{T_i - X_i} \right)^2 - \sum_{i=1}^{n} p_i \left(\frac{A_i}{T_i} \right)^2 \right] \qquad \text{(C9)}$$

Similarly we can write the first and second degree derivatives of F in x_B. Due to symmetry these expressions will be similar to (C8) and (C9).

We can now easily verify the first derivative is zero in every point that satisfies the condition x_A equals x_B (the points along the bisector). We can also prove the second derivative is negative because the two expressions within square parenthesis represent a difference between the square of an arithmetic mean and the arithmetic mean of the squares. This difference is negative because the square of the arithmetic mean is always smaller than the arithmetic mean of the squares.

To prove this result we start from a simple case of two positive numbers n_1 and n_2 and note w_1 and w_2 the weights (obviously w_1 plus w_2 equals 1).We must prove that:

$$w_1 {n_1}^2 + w_2 {n_2}^2 \geq (w_1 n_1 + w_2 n_2)^2 \qquad \text{(C9)}$$

Replacing w_2 with (1- w_1), developing the square parenthesis, moving all factors to the right hand side, and dividing by w_1w_2, we obtain an obvious equivalent inequality:

$$(n_1 - n_2)^2 \geq 0 \tag{C10}$$

We can now use this demonstration to extend the result to arithmetic means of three or more numbers. We can write the arithmetic mean of the squares of three numbers as:

$$w_1 n_1^{\ 2} + w_2 n_2^{\ 2} + w_3 n_3^{\ 2} \geq (w_1 + w_2)\left[\frac{w_1}{(w_1 + w_2)} n_1^{\ 2} + \frac{w_2}{(w_1 + w_2)} n_2^{\ 2}\right] + w_3 n_3^{\ 2} \tag{C11}$$

The expression within square parenthesis is an arithmetic mean of squares and using the previous result we can write:

$$w_1 n_1^{\ 2} + w_2 n_2^{\ 2} + w_3 n_3^{\ 2} \geq (w_1 + w_2)\left[\frac{w_1}{(w_1 + w_2)} n_1 + \frac{w_2}{(w_1 + w_2)} n_2\right]^2 + w_3 n_3^{\ 2} \tag{C12}$$

We can now apply the previous result to obtain that:

$$w_1 n_1^{\ 2} + w_2 n_2^{\ 2} + w_3 n_3^{\ 2} \geq (w_1 n_1 + w_2 n_2 + w_3 n_3)^2 \tag{C13}$$

The second degree derivative of F in x_A is strictly negative unless x_A and x_B are both zero or we have that assets A and B are identical in the sense that:

$$\frac{A_i}{B_i} = const. \text{ for all } i \in [0, n]$$

For the purpose of our demonstration we can ignore both singularities and consider that all second degree derivatives of F are strictly negative.

For every point (x_A, x_B) in the definition space of function F a positive number m exists so that:

$$x_A \le m, \quad x_B \le m, \quad and \ m \in [0,1]$$

We consider the line L that unites the two points (x_A, x_B) and (m,m). This line makes constant angles θ_A and θ_B with both axis x_A and respectively x_B. Both the first and the second derivatives of function F along this line can be written as:

$$F_L^I(x_A x_B) = \cos\theta_A F_A^I(x_{A,} x_B) + \cos\theta_B F_B^I(x_{A,} x_B) \quad \text{(C14)}$$

$$F_L^{II}(x_{A,} x_B) = \cos\theta_A F_A^{II}(x_{A,} x_B) + \cos\theta_B F_B^{II}(x_{A,} x_B) \quad \text{(C15)}$$

We conclude that along line L function F has a unique (and consequently global) local maximum at the point where it meets the bisector. We previously showed in (C5) that function F is constant along the bisector. We can conclude that function F has a local maximum along the bisector, and this maximum is unique (and consequently global). We write:

$$F(x_A, x_B) < F(m,m), \text{ for every } (x_{A,} x_B) \in [0,1] \times [0,1] \text{ and every } m \in [0,1]$$

Reversing the substitutions we defined in (C3), and using (C1) we find that for the points on the bisector we have that:

$$\frac{x_A}{y_A} = \frac{x_B}{y_B} \tag{C16}$$

Unless the investors' portfolios satisfy (C16) the two investors can transact to increase the sum of their portfolios geometric means. Considering (C5) it is obvious that at least a segment on the bisector will further satisfy the condition that the geometric mean of both investors' portfolios will increase following the transaction. We conclude the two investors will transact until they will end up owning the same capital allocation (their portfolios will be similar).

Q.E.D.

Appendix D

In this appendix we calculate equilibrium asset prices. To simplify notations we consider the simple case of two risky assets. We will use the fact that, at equilibrium, each investor holds the market. We can write that:

$$x_A = x_B = c \text{ , where } c \in (0,1) \tag{D1}$$

We use P_A and P_B to denote the equilibrium prices of asset A and B. An investor can exchange x worth of ownership in asset A for x worth of ownership in asset B (or the other way around). Consequently the investor can exchange x / P_A ownership in asset A for x / P_B ownership in asset B (or the other way around). We can define f a function of x representing the geometric expected payoff for the investor's portfolio consequent to the above mentioned transaction. The variable x fully describes the transaction. When x is negative the investor trades ownership in B for ownership in A, when x is positive the investor trades ownership in A for ownership in B. We write:

$$f(x) = \prod_{i=1}^{n} \left[A_i \left(c - \frac{x}{P_A} \right) + B_i \left(c + \frac{x}{P_B} \right) \right]^{p_i} \quad \text{(D2)}$$

The function f must have a maximum in zero. If function f does not have a maximum in zero then a transaction is possible that would further increase the investor portfolio's geometric expected payoff thus contradicting the equilibrium assumption. We extract c outside the product, and operate the substitution $x = \frac{x}{c}$. We can now rewrite (D2) as:

$$f(x) = c \prod_{i=1}^{n} \left[A_i + B_i + x \left(- \frac{A_i}{P_A} + \frac{B_i}{P_B} \right) \right]^{p_i} \quad \text{(D3)}$$

The first derivative of function f is:

$$\frac{\partial f(x)}{\partial x} = f(x) \sum_{i=1}^{n} p_i \frac{\left(- \frac{A_i}{P_A} + \frac{B_i}{P_B} \right)}{\left(A_i + B_i + x \left(- \frac{A_i}{P_A} + \frac{B_i}{P_B} \right) \right)} \quad \text{(D4)}$$

By setting the first derivative in zero equal to zero, eliminating the $f(x)$ factor, and grouping separately P_A and P_B we obtain a linear equation in P_A and P_B:

$$P_A \frac{1}{\sum_{i=1}^{n} p_i \frac{A_i}{M_i}} = P_B \frac{1}{\sum_{i=1}^{n} p_i \frac{B_i}{M_i}} \tag{D5}$$

where $M_i = A_i + B_i$

It must also be the case that the geometric expected payoff for the market as a whole is equal to the total value of the market (since the expected cumulative return must be equal to time value of money and consequently the geometric expected return for the market as a whole must be equal to one). We write:

$$GM = P_A + P_B \tag{D6}$$

Equations (D5) and (D6) form a system of two linear equations in P_A and P_B. This system has a unique set of solutions:

$$P_A = GM \sum_{i=1}^{n} p_i \frac{A_i}{M_i} \text{, and } P_B = GM \sum_{i=1}^{n} p_i \frac{B_i}{M_i} \tag{D7}$$

The geometric expected return is given by (8). The arithmetic expected return of asset A is given by:

$$AR_{A^j} = \frac{1}{GM} \cdot \frac{AM(A_i)}{AM(P_i)} \tag{D8}$$

where

$$AM(A_i) = \sum_{i=1}^{n} p_i A_i$$

We must note that all the above calculations remain valid no matter how many risky assets exist in the market (we considered the case of two assets to simplify notations).

As an example, we can use (D7) to calculate the equilibrium prices of asset A and B described in Table 1 (Appendix C). We find that the equilibrium price of asset A would be $86.61 and the equilibrium price of asset B would be $134.95.

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FOOTNOTES

[1] Von Neumann and Morgenstern (1944) proposed axioms describing rational choice under uncertainty and then showed the utility function can be taken to be linear in the probabilities.

[2] See Black, Jensen and Scholes (1972), Blume and Friend (1972), and Fama and MacBeth (1973).

[3] See Markowitz (1959), and Tobin (1958).

[4] The development of the CAPM was accomplished by Sharpe (1964) and Lintner (1965) on the foundation laid down by Markowitz (1959) and Tobin (1958) with their studies on mean variance choice.

[5] Since the expected cumulative return form owning the market must be equal to T (the discount rate describing time value of money) it follows that the geometric expected return of the market must be equal to one. Consequently the total value of the market must equal its geometric expected payoff.

TABLES

Assets / States	A	B	Probabilities
1	$100	$10	0.1
2	$120	$80	0.5
3	$0	$300	0.4

Table I Probability distribution of payoffs

The table describes a simple world with two risky assets and three possible states. Columns A and B indicate an asset's payoffs in each particular state. Rows 1, 2, and 3 indicate all the asset payoffs in a given state. The last column indicates the probabilities for each state.

FIGURES

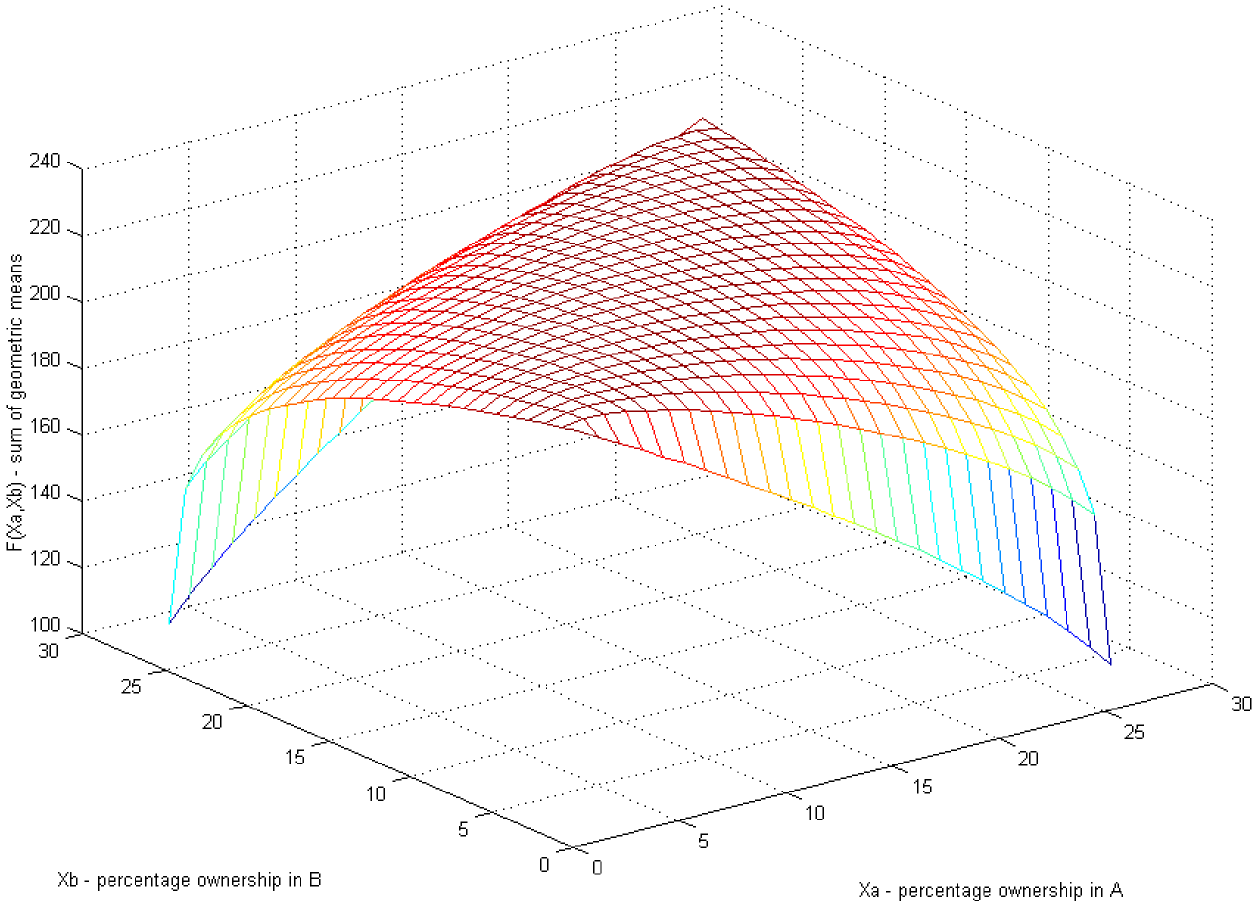

Figure 1 3D graph of function F

The figure illustrates the sum of the geometric expected payoffs of the two investors as a function of their capital allocation. Xa and Xb are the percentage ownership of investor 1 in asset A and respectively asset B. Since investor 2 owns the remaining portion of the two assets, the pair Xa and Xb entirely describes the capital allocation of the two investors. We notice that the sum of the geometric expected payoffs is maximized when the two investors own the market.

