Looking Forward to Pricing Options from Binomial Trees

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Abstract

We reconsider the valuation of barrier options by means of binomial trees from a "forward looking" prospective rather than the more conventional "backward induction" one used by standard approaches. Our reformulation allows us to write closed-form expressions for the value of European and American put barrier-options on a non-dividend-paying stock.

I. INTRODUCTION

Closed-form valuation within the Black-Scholes-Merton equilibrium pricing theory [1,2] is only possible for a small subset of financial derivatives. In the majority of cases one must appeal to numerical techniques such as Monte Carlo simulations, or finite difference methods and much of the effort in the field has been in developing efficient algorithms for numerically solving the Black-Scholes equation [3]. An alternative direction has been the evaluation of discrete-time, discrete-state stochastic models of the market on binomial and trinomial trees [4,5]. Not only is this discrete approach intuitive and easily accessible to a less mathematically sophisticated audience; but it also seems to us to be a more accurate description of market dynamics and better suited for evaluating more involved financial instruments. Moreover, the few exact Black-Scholes results available can be recovered in the appropriate continuous-time trading limit. The main difficulty in pricing with binomial trees has been the non-monotonic numerical convergence and the dramatic increase in computational effort with increasing number of time steps [6,7]. For example, the state of the art calculations involve memory storage scaling linearly (quadratically) with the number of time steps, N, for European (American) options, while the computation time increases like N^2 in both cases [8].

In this paper we reconsider valuation on binomial trees from what we call a "forward looking" prospective: we imagine acting as well-educated consumers who attempt to eliminate risk and estimate the value of an option by looking at its future expected values according to some reasonable dynamical model. As will be described in great detail below the main intuitive idea of our computations is to regard the movement of the price on the tree as a random walk (with statistical properties consistent with a risk-neutral world) with "walls" imposed by the nature of the option, such as the possibility of early exercise (American options) or the presence of barriers. Our mathematical formulation then has two conceptually distinct components: the first ingredient is an explicit description of the possible "walls". For example, in the case of barrier American options both the barrier and the "early exercise" surface need to be specified. The second step will be to compute the probability that the price reaches particular values at every accessible point on the tree. This involves counting the number of paths reaching that point in the presence of "walls", a somewhat involved but exactly solvable combinatorics problem. Once these two steps (specifying the walls and computing the probabilities) are accomplished the value of both European and American options, with and without barriers, can be written down explicitly. In an attempt to be pedagogical, we will limit ourselves to the simplest cases: we will only treat European, simple American and European with a straight "up-and-out" barrier. Although the calculation can be simply extended to the barrier American option that discussion merits a separate publication.

We note that, as far as we know, in the case of trees explicit formulas like the ones we are proposing exist in the literature only in the simplest case of conventional European options [5,9]. For the more complicated case of American options, the main issues are best summarized in the last chapter of Neil Chriss' book [10]: "The true difficulty in pricing American options is determining exactly what the early exercise boundary looks like. If we could know this *a priori* for any option (e.g., by some sort of formula), we could produce pricing formulas for American options." Below we propose a solution to this problem in the

context of binomial trees. Our formulation complements the earlier studies of American options in the limit of continuous-time trading [11–13] which also focus on the presence of an early exercise boundary for the valuation of path-dependent instruments. The study of the continuum limit of our formulas is instructive and will be left for a future publication.

II. BINOMIAL TREES

To establish notation we begin by dividing the life of an option, T, into N time intervals of equal length, $\tau = T/N$. We assume that at each discrete time $t_i = i\tau$ $(i = 0, 1, 2, ..., N)$ the stock price moves from its initial value, $S_0 = S(t_0 = 0)$, to one of two new values: either up to $S_0 u$ $(u > 1)$ or down to $S_0 d$ $(d < 1)$ [8]. This process defines a tree with nodes labeled by a two dimensional vector, (i, j) $(i = 0, 1, 2, ..., N; j = 0, 1, ..., i)$ and characterized by a stock price $S(i, j) = S_0 u^j d^{i-j}$, the price reached at time $t_i = i\tau$ after j up and $i - j$ down movements, starting from the original price S_0. The probability of an up (down) movement will be denoted by p_u $(p_d = 1 - p_u)$; and thus each point on the tree is also characterized by the probability, $p_u^j (1 - p_u)^{i-j}$, which represents the probability associated with a single path of i time steps, j $(i-j)$ of which involve an increase (decrease) in the stock price. Computing the probability of connecting the origin with point (i, j) requires, in addition to the single path probability, a factor counting *the number* of such possible paths in the presence of a barrier and/or the possibility of early exercise. The calculation of this degeneracy factor involves the details of each financial derivative and it will be discussed in turn for each of our examples.

The binomial tree model introduces three free parameters, u, d and p_u. Two of these are usually fixed by requiring that the important statistical properties of the random process defined above, such as the mean and variance, coincide with those of the continuum Black-Scholes-Merton theory [3]. In particular,

$$p_u u + (1 - p_u) d = e^{r\tau} \tag{1}$$

$$e^{r\tau} (u + d) - ud - e^{2r\tau} = \sigma^2 \tau, \tag{2}$$

where r is the risk-free interest rate, and the volatility, σ, is a measure of the variance of the stock price. We are left with one free parameter which can be chosen to simplify the theoretical analysis; one might choose, for example, $u = 1/d$ [4], which simplifies the tree geometry by arranging that an up motion followed by a down motion leads to no change in the stock price. This condition together with (**??**) and (1) imply:

$$u = e^{\sigma\sqrt{\tau}} \tag{3}$$

$$d = e^{-\sigma\sqrt{\tau}} \tag{4}$$

$$p_u = \frac{e^{r\tau} - d}{u - d}. \tag{5}$$

We stress that Equations (1-5) are to be regarded as short-time approximations where terms higher order in τ were ignored.

With these definitions out of the way we can begin discussing the valuation of put options with strike price X and expiration time T.

A. European Put Options

The simple European put option is a good illustration of our "forward looking" approach. We are interested in all those paths on the tree which, at expiration time $i = N$, reach a price, $S(N, j) = S_0 u^j d^{N-j} < X$, for which the option should be exercised. That implies that $j \leq j^* = \text{Int}\left[\ln(X/S_0 d^N)/\ln(u/d)\right]$, where Int refers to the integer part of the quantity in square brackets. The mean value of the option at expiration can then be written as a sum over all values of $j \leq j^*$ of the payoff at j, $X - S_0 u^j d^{N-j}$, multiplied by the probability of realizing the price $S(N, j) = S_0 u^j d^{N-j}$ after N time steps, $P[N, j]$. As already mentioned above, $P[N, j] = \aleph_E[N, j] p_u^j (1 - p_u)^{N-j}$, where $\aleph_E[N, j]$ counts the number of paths starting at the origin and reaching the price $S(N, j)$ in N time steps. For the case of conventional European options this is just the number of paths of N time steps, with j up and $N - j$ down movements of the price, and is thus given by the binomial coefficient,

$$\aleph_E[N, j] \;=\; \left(\begin{array}{c} N \\ j \end{array} \right) \;=\; \frac{N!}{j!(N-j)!}. \tag{6}$$

The resulting expression for the mean value of the option at maturity is then discounted to the time of contract by the risk-free interest rate factor, e^{-rT}, to determine the current expected value of the option:

$$\bar{V}_E = e^{-rT} \sum_{j=0}^{j^*} \left(\begin{array}{c} N \\ j \end{array} \right) p_u^j (1 - p_u)^{N-j} \left(X - S_0 u^j d^{N-j} \right). \tag{7}$$

This expression is not new: it was first discussed by Cox and Rubinstein [5] who also showed that in the appropriate continuous trading-time limit ($\tau \to 0$) (7) reduces to the Black-Scholes result [1].

B. European Put Barrier Options

We are now ready to extend (7) into an exact formula for the mean value of an European put option with a barrier. Although our approach can be used for other barrier instruments, we consider the simplest case of an "up-and-out" put option which ceases to exist when some barrier price, $H > S_0$, higher than the current stock is reached. With the choice $u = 1/d$ an explicit equation for the nodes of the tree which constitute the barrier can be written down:

$$S(j_B + 1 + 2h, j_B + 1 + h) = S_0 u^{j_B + 1 + h} d^h, \qquad h = 0, 1, ..., h_B \tag{8}$$

Here, $j_B = \text{Int}\left[\ln\left(H/S_0\right)/\ln(u)\right]$ defines the first point just above the barrier, $(j_B{+}1, j_B{+}1)$, and h_B labels the last relevant point on the barrier corresponding to the time closest to the maturity of the option, i.e., $h_B = \text{Int}\, t[(N - j_B - 1)/2]$.

Since the probability that any allowed path starting with the present stock price, S_0, reaches an exercise price at maturity, $S(N, j) < X$, is still $p_u^j (1 - p_u)^{N-j}$ (with $j \leq j^*$) the average value of the European barrier option can be written in a form similar to (7):

$$\bar{V}_{EB} = e^{-rT} \sum_{j=0}^{j^*} \aleph_{EB}[N, j]\; p_u^j (1 - p_u)^{N-j} \left(X - S_0 u^j d^{N-j} \right), \tag{9}$$

where $\aleph_{EB}[N,j]$ is the number of paths N time-steps long involving j up and $N-j$ down movements of the price excluding those paths reaching any of the points on or above the barrier (8). As we will explain below, $\aleph_{EB}[N,j]$ is given by

$$\aleph_{EB}[N,j] \;=\; \left(\begin{array}{c} N \\ j \end{array}\right) \;-\; \sum_{h=0}^{h_M} \aleph^{res}_{EB}[j_B+1+2h, j_B+1+h] \left(\begin{array}{c} N-j_B-1-2h \\ j-j_B-1-h \end{array}\right), \qquad (10)$$

where the second term on the right-hand side represents the contribution from the unwanted paths which hit the barrier (8) before reaching an exercise point (N,j).

To understand the form of the excluded contribution in (10) we first note that reaching the excluded region requires that the path hits the barrier at least once. Thus in counting the excluded paths we can simply focus on those paths reaching the barrier. One might think that the number of unwanted paths can then be calculated by (i) counting the number of paths connecting the origin to a given point on the barrier; (ii) multiplying this by the number of paths connecting that point on the barrier with the exercise point (N,j) [14]; and finally (iii) summing over all points of the barrier (8). This seems correct except that all paths connecting a barrier point with (N,j) already include those paths which have already hit the barrier once or more before reaching the given barrier point. To eliminate overcounting we must make sure that in (i) we only include paths reaching the particular point on the barrier without having previously visited any other point on the barrier. The number of such restricted paths (reaching the point (j_B+1+2h, j_B+1+h)) is denoted by $\aleph^{res}_{EB}[j_B+1+2h, j_B+1+h]$. Finally note that the final sum over the length of the barrier is restricted to $h \le h_M = min(h_B, j-j_B-1)$ with $j \ge j_B+1$, corresponding to the fact that, in general, the exercise point (N,j) cannot be reached from all points on the barrier. This completes our explanation of (10).

We are then left with computing $\aleph^{res}_{EB}$. From its very definition it is not hard to see that $\aleph^{res}_{EB}[h] \equiv \aleph^{res}_{E}[j_B+1+2h, j_B+1+h]$ satisfies the following recursion relation:

$$\aleph^{res}_{EB}[0] \;=\; 1 \qquad (11)$$

$$\aleph^{res}_{EB}[h] \;=\; \left(\begin{array}{c} j_B+1+2h \\ j_B+1+h \end{array}\right) \;-\; \sum_{l=0}^{h-1} \aleph^{res}_{EB}[l] \left(\begin{array}{c} 2(h-l) \\ h-l \end{array}\right), \qquad h \ge 1, \qquad (12)$$

with the sum in (11) removing contributions from previously visited barrier points. Obviously $\aleph^{res}_{EB}[0] = 1$ as there is a single path involving j_B+1 up moves connecting the origin with the point (j_B+1, j_B+1) on the tree.

To solve Equations (**??**) and (11) we first combine the sum on the right-hand side of (11) with the term on the left and rewrite the resulting equation in the form of a discrete convolution:

$$\sum_{l=0}^{h} \aleph^{res}_{EB}[l] \left(\begin{array}{c} 2(h-l) \\ h-l \end{array}\right) = \left(\begin{array}{c} j_B+1+2h \\ j_B+1+h \end{array}\right), \qquad (13)$$

where the boundary condition, $\aleph_{EB}[0] = 1$, is already included as the $h=0$ contribution to (13). Note that (13) can be solved by standard Laplace transform (or Z-transform) techniques [15]. Since in applying these ideas to the more complicated American options we will lose the convolution form of (13) – the kernel will depend on h and l separately and

not only through the difference, $h-l$ – we prefer to proceed in a more general way and stay in "configuration space" until the very end. We prefer to regard (13) as a matrix equation to be solved by matrix inversion.

We proceed by reformulating (13) in the following matrix form:

$$\mathbf{L}_{EB}\mathbf{\Pi}^{res}_{EB} = \mathbf{D}_{EB}, \tag{14}$$

where $\mathbf{\Pi}_{EB}$ and $\mathbf{D}_{EB}$ are h_M+1 dimensional vectors, with components $\mathbf{\Pi}_{EB,h} = \aleph^{res}_{EB}[h]$ and $\mathbf{D}_{EB,h} = \begin{pmatrix} j_B+1+2h \\ j_B+1+h \end{pmatrix}$, $h = 0, 1, 2, ..., h_M$, and the $(h_M+1)\times(h_M+1)$ dimensional matrix, $\mathbf{L}_{EB}$, can be written as,

$$[\mathbf{L}_{EB}]]_{h,l} \;=\; \begin{pmatrix} 2(h-l) \\ h-l \end{pmatrix} \theta\,(h-l)\,. \tag{15}$$

Note that in (15) we have explicitly added a θ function ($\theta(x) = 1$ for $x \geq 0$ and vanishes otherwise) to stress that $\mathbf{L}_{EB}$ is a lower triangular matrix with unity along and zeros above the diagonal. This simple observation allows us to rewrite (14) in the convenient form,

$$\mathbf{L}_{EB} = \mathbf{1}_{(h_M+1)\times(h_M+1)} + \mathbf{Q}_{EB}, \tag{16}$$

where $\mathbf{Q}_E$ is a nilpotent matrix of order h_M, $\mathbf{Q}_{EB}{}^y = 0$ for $y \geq h_M+1$; and $[\mathbf{Q}_{EB}]]_{h.l} = \begin{pmatrix} 2(h-l) \\ h-l \end{pmatrix}\theta(h-l-1)$. The nilpotent property of $\mathbf{Q}_{EB}$ allows us to write down the explicit solution for (14),

$$\mathbf{\Pi}_{EB} = [\mathbf{1} \;+\; \mathbf{Q}_{EB}]^{-1}\,\mathbf{D}_{EB} \;=\; \sum_{R=0}^{h_M}(-1)^R\mathbf{Q}^R_{EB}\mathbf{D}_{EB}, \tag{17}$$

which in turn leads to the following formula for the value of the option,

$$\bar{V}_{EB} = \bar{V}_E - \bar{V}^{res}_{EB} \tag{18}$$

$$\bar{V}^{res}_{EB} = e^{-rT}\sum_{j=j_B+1}^{j^*}\sum_{h,l,R=0}^{h_M(j)}(-1)^R\begin{pmatrix} N-j_B-1-2h \\ j-j_B-1-h \end{pmatrix}\left[\mathbf{Q}^R_{EB}\right]]_{h,l}\begin{pmatrix} j_B+1+2l \\ j_B+1+l \end{pmatrix} \tag{19}$$
$$\times\, p_u^j(1-p_u)^{N-j}\left(X - S_0u^jd^{N-j}\right).$$

Note that the lower limit on the external sum, $j = j_B+1$, excludes all paths unaffected by the presence of the barrier; also we have explicitly indicated the N and/or j dependence of the various quantities involved; and have separated out the contribution to $\aleph_{EB}[N,j]$ from unrestricted paths (the first term on the right-hand side of (10)) which simply leads to the value of the European put option given in (7).

We expect that, since we have an analytical formula, we should be able to recover the exact solution of the continuum Black-Scholes theory for this simplest of barrier options [16] as was already done for conventional European puts [5]. Figure 1 shows the numerical convergence of the binomial value of a representative "up-and-out" European put option to its analytic value [16].

The same general idea used in the case of European barrier options will now be used to write down an exact formula for the price of a simple American option, regarding the latter as an option with an early-exercise barrier.

C. Conventional American Put Options

Using this view to valuate American options requires the knowledge of those points on the tree where it first becomes profitable to exercise the option. This set of points, parameterized as $(i, j_x[i])$, constitute the "early exercise barrier" (EXB). Determining the explicit form of the surface, $j_x[i]$, seems very difficult (if at all possible) as it already implies a knowledge of the mean value of the option at some finite number of points on the tree. In this section we show that there is a self-consistent exact formulation of the problem which proceeds in the following three steps: (i) we assume that the early exercise surface, $j_x[i]$ is given and compute an explicit formula for the value of the option at each point on the tree, $f(i, j; j_x[i])$, which depends parametrically on $j_x[i]$; (ii) the fact that early exercise at (i, j) only occurs when $X - S_0 u^j d^{i-j} \geq f(i, j; j_x[i])$ gives us an explicit formula for the EXB which corresponds to the strict equality,

$$\left(X - S_0 u^{\tilde{j}[i;j_x[i]]} d^{i-\tilde{j}[i;j_x[i]]}\right) \;=\; f(i, j_x[i]); \qquad j_x[i] = \text{Int}\left\{\tilde{j}[i; j_x[i]\right\}. \tag{20}$$

[Note that, on the right-hand side of (20) we have not used $f(i, \tilde{j}[i])$ which might appear at first sight as a more natural choice for defining the EXB. As will become clear below, (20) is the simplest and most natural choice which resolves the ambiguity of defining $f(i, j)$ away from points on the tree.] Finally, (iii) substituting the solution (20) into the formally exact valuation expression gives us the value of the option. Although this strategy leads to an exact solution of the price of an American option, explicit numbers require rather heavy numerical computations except in the simplest example of a straight EXB.

Let us proceed in carrying out the program outlined above by assuming that the EXB, i.e., $j_x[i]$, is explicitly given. To begin our calculation we will need some very general properties of the barrier. These follow from two simple characteristics of early exercise: (i) if the point (i, j) is an early exercise point, then so are all points "deeper in-the-money", (i, j'), $j' = 0, 1, ..., j-1$; and (ii) if two adjacent points at the same time step, $(i+1, j+1)$ and $(i+1, j)$, are both early exercise points so is the point (i, j). (The latter property follows from a conventional "backwardation" argument [3] which indicates that the average expected payoff at (i, j), discounted at the risk-free interest rate, is smaller than the actual payoff, thus making (i, j) itself an early exercise point.) It is not hard to see that (i) and (ii) guarantee that the inner part of the early exercise region cannot be reached without crossing the EXB. Thus, if we define i_A to be the first time for which early exercise becomes possible and parametrize the points on the EXB as $(i = i_A + h, j_x[i_A + h])$ with $h = 0, 1, 2, ..., N - i_A$, it then follows that $j_x[i_A] = 0$. Moreover, the structure of the tree ensures that $j_x[i]$ is a nondecreasing function of i; more precisely, for each time step, $j_x[i]$ either increases by one or remains the same.

The formal expression for the price of an American option can be written down once one recognizes that once a path hits the EXB the option expires and thus any point on the barrier can be reached at most once. As a result, the value of the option is a sum of (appropriately discounted) payoffs along the barrier, weighted by the probability of reaching each point on the barrier without having visited the barrier at previous times. We can then write the expected value of an American option as:

$$\bar{V}_A = \sum_{h=0}^{N-i_A} e^{-r(i_A+h)\tau} \aleph_A^{res}[h] p_u^{j_x[i_A+h]} (1-p_u)^{i_A+h-j_x[i_A+h]} \left(X - S_0 u^{j_x[i_A+h]} d^{i_A+h-j_x[i_A+h]} \right), \tag{21}$$

where $\aleph_A^{res}$ denotes the number of paths reaching the EXB in $i_A + h$ time steps without having previously visited any points on the barrier.

The counting problem can be solved along similar lines to those followed in the case of European options: $\aleph_A^{res}[h]$ satisfies an equation analogous to (11), namely,

$$\aleph_A^{res}[0] = 1 \tag{22}$$

$$\aleph_A^{res}[h] = \begin{pmatrix} i_A + h \\ j_x[i_A + h] \end{pmatrix} - \sum_{l=0}^{h-1} \aleph_A^{res}[l] \begin{pmatrix} (h-l) \\ j_x[i_A + h] - j_x[i_A + l] \end{pmatrix}, \qquad h \geq 1, \tag{23}$$

where the first term on the right-hand side counts the total number of unrestricted paths from the origin to the point $(i_A + h, j_x[i_A + h])$ on the barrier, while the second term excludes those paths which, before reaching $(i_A + h, j_x[i_A + h])$ visited any of the previous barrier points, $(i_A + l, j_x[i_A + l])$, $l = 0, 1, 2, ..., h - 1$ [14].

As in the case of the European barrier option (22) is rewritten as a matrix equation:

$$\mathbf{L}_A \mathbf{\Pi}_A^{res} = \mathbf{D}_A. \tag{24}$$

Here $\mathbf{\Pi}_A$ and $\mathbf{D}_A$ are $N - i_A + 1$ dimensional vectors, with components $\mathbf{\Pi}_{A,h} = \aleph_A^{res}[h]$ and $\mathbf{D}_{A,h} = \begin{pmatrix} i_A + h \\ j_x[i_A + h] \end{pmatrix}$, $h = 0, 1, 2, ..., N - i_A$, and the $(N - i_A + 1) \times (N - i_A + 1)$ dimensional matrix, $\mathbf{L}_A$, takes the form,

$$[\mathbf{L}_A]_{h,l} = \begin{pmatrix} h-l \\ j_x[i_A + h] - j_x[i_A + l] \end{pmatrix}, \quad l \leq h = 0, 1, 2, ..., N - i_A. \tag{25}$$

Note that, in contrast to (15) and (16), $\mathbf{L}_A$ depends on the indices h and l separately; also, we have used the identities $j_x[i_A] = 0$ and $\begin{pmatrix} i_A \\ j_x[i_A] \end{pmatrix} = 1$, to incorporate the boundary condition, $\aleph_A^{res}[0] = 1$, in (24) in a symmetric way. As in (16), we can decompose $\mathbf{L}_A$ as,

$$[\mathbf{L}_A]]_{h,l} = \delta_{h,l} + [\mathbf{Q}_A]_{h,l} \tag{26}$$

$$[\mathbf{Q}_A]_{h,l} = \begin{pmatrix} h-l \\ j_x[i_A + h] - j_x[i_A + l] \end{pmatrix} \theta(h - l - 1), \tag{27}$$

where $\mathbf{Q}$ has nonzero elements starting just below the diagonal and it is thus a nilpotent matrix of degree $N - i_A + 1$. Thus,

$$\mathbf{\Pi}_A = [\mathbf{1} + \mathbf{Q}_A]^{-1} \mathbf{D}_A = \sum_{m=0}^{N-i_A} (-1)^m \mathbf{Q}_A^m \mathbf{D}_A, \tag{28}$$

leading in turn to the final formula for the value of the option,

$$\bar{V}_A = \sum_{h,l,m=0}^{N-i_A} e^{-r(i_A+h)\tau}(-1)^m \left[\mathbf{Q}_A^m\right]_{h,l} \begin{pmatrix} i_A+l \\ j_x[i_A+l] \end{pmatrix} \times \quad p_u^{j_x[i_A+h]}(1-p_u)^{i_A+h-j_x[i_A+h]} \left(X - S_0 u^{j_x[i_A+h]} d^{i_A+h-j_x[i_A+h]}\right). \tag{29}$$

One last step is the determination of $f(i,j)$, the value of the American put at every point (i,j) on the tree which, in turn, will allow us to derive the equation for the EXB. This is easily done by simply translating the origin in (29):

$$f(i,j) = \sum_{h,l,m=i-i_A}^{N-i_A} e^{-r(i_A+h-i)\tau}(-1)^m \left[\mathbf{Q}_A^m\right]_{h,l} \begin{pmatrix} i_A+l-i \\ j_x[i_A+l]-j \end{pmatrix} \times p_u^{j_x[i_A+h]-j}(1-p_u)^{i_A+h-j_x[i_A+h]-i+j} \left(X - S_0 u^{j_x[i_A+h]} d^{i_A+h-j_x[i_A+h]}\right). \tag{30}$$

Together with (20) this then leads to the rather formidable-looking equation for the barrier height $\tilde{j}[i_A+k]$ at the (i_A+k)-th time step $(k=0,1,...,N-i_A)$, as a functional of the barrier position at all *future* time steps before expiration:

$$\begin{aligned} \left(X - S_0 u^{\tilde{j}[i_A+k]} d^{i_A+k-\tilde{j}[i_A+k]}\right) &= \sum_{h,l,m=k}^{N-i_A} e^{-r(h-k)\tau}(-1)^m \left[\mathbf{Q}_A^m\right]_{h,l} \begin{pmatrix} l-k \\ j_x[i_A+l]-j_x[i_A+k] \end{pmatrix} \\ &\quad \times p_u^{j_x[i_A+h]-j_x[i_A+k]}(1-p_u)^{h-k-j_x[i_A+h]+j_x[i_A+k]} \\ &\quad \times \left(X - S_0 u^{j_x[i_A+h]} d^{i_A+h-j_x[i_A+h]}\right) \qquad (31) \\ j_x[i_A+k] &= \mathrm{Int}\left\{\tilde{j}[i_A+k]\right\}. \qquad (32) \end{aligned}$$

[It should now be clear that in (30) j must be restricted to points on the tree as the binomial coefficient $\begin{pmatrix} 0 \\ j_x[x_A+l]-\tilde{j}[x_A+l] \end{pmatrix}$ would be ill-defined – hence the choice (20).] Equations (**??**) and (31) for the boundary together with the formula for the value of the option, (29), constitute an exact pricing strategy for a conventional American put. A similar formula for an American put with an "up-and-out" barrier will be discussed in a future publication.

It is instructive to consider Equations (29), (**??**) and (31) in the explicitly solvable case of a straight barrier. We begin with the observation that at expiration, $k = N-i_A$, (**??**) reduces to the definition of $j^* = \mathrm{Int}[\ln(X/S_0 d^N)/\ln(u/d)]$, defined in the case of the European option, and thus, the barrier goes through the point (N, j^*) at expiration. Moreover, starting from the exact point (N, j^*) on the barrier and decreasing $j_x[i]$ by one with each backward time step we reach $i_A = N - j^*$ along the straight line, $j_x[i] = i - N + j^*$. Recall that, since with each increasing time step, $j_x[i]$ either increases by one or remains the same, this straight line represents a lower bound for the early exercise barrier.

For this straight barrier (28) and (29) reduce to,

$$\begin{aligned} P_A^{straight} &= e^{-r(N-j^*)\tau}(1-p_u)^{N-j^*}\left(X - S_0 d^{N-j^*}\right) \\ &\quad + \sum_{h=1}^{j^*} e^{-r(N-j^*+h)\tau} \aleph_A^{straight}[h]\; p_u^h (1-p_u)^{N-j^*}\left(X - S_0 u^h d^{N-j^*}\right) \end{aligned} \tag{33}$$

with

$$\aleph_A^{straight} = \begin{pmatrix} N - j^* + h \\ h \end{pmatrix} - \begin{pmatrix} N - j^* + h - 1 \\ h - 1 \end{pmatrix} \tag{34}$$

We expect that the result for the true barrier should approach the straight line formula for coarse enough time steps, $\tau > N - j^* - i_A$, (where this i_A is the first time of early exercise in the limit of continuous-time trading).

III. CONCLUSION

We have presented a scheme for pricing options with and without barriers on binomial trees. To the best of our knowledge ours is the first explicit derivation of exact formulas treating barriers on binomial trees. It is our expectation that in the limit of continuous-time trading we should be able to recover the few exact results available in the literature, especially for American options [12,13]. We also hope that our explicit formulas may provide a framework for improving the efficiency of numerical computations.

IV. ACKNOWLEDGEMENTS

The authors dedicate this paper to Professor Ferdinando Mancini, a remarkable teacher, colleague and friend, on the occasion of his 60th birthday. We are grateful to Stanko Barle for reading the manuscript and bringing the work of references [12] and [13] to our attention.

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[14] We recall that the number of paths between two arbitrary points on the tree, say (i, j) and (i', j') (with $i' > i$, $j' > j$) is given by the binomial coefficient, $(i'-i)!/[(j'-j)!(i'-i-j'+j)!]$.

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FIGURES

FIG. 1. Convergence to analytic value [16] of a three-month European "up-and-out" put option on a non-dividend-paying stock as the number of binomial intervals N increases. The solid line joining the squares is a guide to the eye. The stock price S_0 is 60, the risk-free interest rate r is 10%, and the volatility σ is 45%. The barrier level H is at 64. The analytic value (continuous line) is 2.524. As a point of reference, we give the European put option value (i.e., $H \to \infty$) 4.6 after the Black-Scholes solution (dashed line).