Best Log-linear Index Numbers: Extensions and Applications

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Abstract. Index numbers of prices and quantities are estimated in the framework of a two-way analysis of variance, based on the ideas of H. Theil and K. Banerjee. Topics include aggregation, multidimensional indices, and non-spherical errors. The method is applied to the commodity exports of developing nations.

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Introduction

Price and quantity indices have usually been treated as descriptive statistics. However, Blankmeyer (1990) showed that standard econometric methods can be applied to estimate index numbers and to test hypotheses about rates of change in prices and quantities. This shift from description to inference is based on three ideas.

First, index numbers should be derived using all the sample observations jointly. In the conventional approach, a "base period" is selected more or less arbitrarily. The remaining observations are then compared to the base period one at a time. This pairwise formula makes little sense if the construction of indices is approached as an estimation problem.

Second, there can be no unique estimate of the actual level of a price or quantity index. Only rates of change are identifiable since there is one degree of freedom in scaling each index.

Third, prices and quantities are to be treated symmetrically. For the price index, the weights should be computed from all the quantity data, while all the price data should be taken into account in weighting the quantity index.

No doubt these ideas have been advocated more than once in the vast literature on index numbers. From this author's viewpoint, they are a synthesis of several papers that appeared more than thirty years ago. Henri Theil (1960) made a strong case for indices derived symmetrically and jointly from all the observations. His eigenvalue formulas are "closely related to principal-component analysis....There is a difference from the usual type of principal-component analysis in that we consider here two sets of variables, viz., prices and quantities" (p. 465).

This remark suggests that Theil's indices are also related to canonical correlation; indeed, they maximize a zero-order moment in the two sets of variables. Interestingly, Gerhard Tintner (1952) applied canonical correlation explicitly to index numbers. However, he adopted a curious approach, in effect multiplying the price of apples by the quantity of oranges ! Most researchers, including Theil, have preferred to multiply the price of apples in one period by the quantity of apples in another period; they then repeat the process for oranges and aggregate the results. This is the approach taken in the present paper.

Kali Banerjee pointed out that index numbers fit naturally into the format of a two-way analysis of variance (ANOVA). He developed this least-squares treatment for the pairwise model (base year/current year) in a series of papers during the 1950s and 1960s. They are summarized in two monographs [Banerjee (1975,1977)]. Subsequently, D. S. Prasada Rao, Banerjee and others (1986,1995) extended the ANOVA method to handle multinational price indices. This is the spatial counterpart of the idea that all time periods should be considered jointly.

In their seminal papers, these researchers continued to treat index numbers as descriptive statistics. They do not seem to have formulated an estimation problem, nor did they propose hypothesis tests. However, their methods lead to the best log-linear (BLL) index numbers, which are briefly reviewed in the next section.

The present paper has several objectives. It discusses concisely how the BLL method can handle problems of aggregation, multiple factors, and nonspherical errors. Next, some of these concepts are applied to exports of raw materials from developing nations between 1976 and 1985. The paper concludes with remarks on the "statistical" and "economic" approaches to index numbers.

BLL Indices

Multiperiod index numbers are based on the hypothesis of equiproportional variation. If, from one period to the next, all commodity prices changed in lockstep, and if the corresponding quantities also rose or fell in tandem, then the computation of indices would be straightforward. However, allowance must be made for random departures from lockstep. Given T joint observations on K commodities, let $\exp(v_{rt}) = \Sigma_k P_{rk} Q_{tk}$ denote the aggregate value when the prices of period r are applied to the quantities of period t. The validity of equiproportional variation can be tested in the log-linear model:

$$v_{rt} = p_r + q_t + z + e_{rt}\,, \qquad r, t = 1, ..., T. \qquad (1)$$

The period-r log price index, p_r, and the period-t log quantity index, q_t, are unknown parameters that must be estimated, as is the intercept, z. The unobserved errors, e_{rt}, account for the failure of prices and quantities to change in lockstep. If e_{rt} conforms to the assumptions of the Gauss-Markov theorem, then application of ordinary least squares (OLS) to equation (1) leads to the two-way ANOVA model.

It is well known that the "independent variables" for that model are a set of dummies (0's and 1's) subject to two linear dependencies [F. A. Graybill (1976), ch. 14]. As stated earlier, these dependencies simply mean that the levels of the price and quantity indices cannot be determined uniquely. Only the log-linear contrasts, p_r-p_t and q_r-q_t are identifiable; they measure the percent change in the levels of their respective indices.

Under the arbitrary but convenient normalizations $\Sigma p_r = \Sigma q_t = 0$, application of OLS produces the familiar two-way ANOVA solution:

$$z = \Sigma\Sigma v_{rt}/T^2,$$

$$p_r = \Sigma_t v_{rt}/T - z, \text{ and} \qquad (2)$$

$$q_t = \Sigma_r v_{rt}/T - z,$$

The residuals

$$e_{rt} = v_{rt} - p_r - q_t - z \qquad (3)$$

are in turn the basis for an unbiased estimate of the error variance:

$$s^2 = \Sigma\Sigma e_{rt}^2/(T-1)^2 \qquad (4).$$

If it can be assumed that the unobserved errors are spherical normal variables, then hypothesis tests are readily available. Blankmeyer (1990) gives several examples. For instance, the statistic

$$(p_r - p_t - \pi) / (2s^2/T)^{1/2} \qquad (5)$$

has Student's distribution with $(T-1)^2$ degrees of freedom on the null hypothesis that the percent change in the price level was π between periods r and t.

Of course, the assumption of spherical normal errors is fairly drastic and may have to be modified. A model with heteroscedasticity is sketched in the sequel. Blankmeyer (1991) outlines a robust alternative to OLS for error distributions that are more diffuse than the normal.

In deference to the descriptive approach to index numbers, it may be mentioned that the BLL method is an extension of Irving Fisher's ideal indices. In fact, for T = 2, $\exp(p_2-p_1)$ is the ideal price index. Unlike conventional (base year/current year) formulas, the BLL indices meet the circularity test for all periods in the sample and therefore provide consistent estimates of rates of change over three or more time intervals.

It is worth noting that the BLL model is not limited to analyzing time series. The indices r and t may refer to pairs of cities, nations, or industries. The index numbers may measure the productivity and utilization of resources rather than the prices and quantities of goods.

Aggregation.

Often data are available to compute price and quantity indices for several groups of commodities --for example, food (f) and clothing (c). An important question is how such indices should be combined to form an aggregate (a). In the BLL format, three separate regression equations can be estimated --one each for f, c, and a. It is no surprise that, in the absence of restrictions across the equations, the indices for food and clothing do not add up to the aggregate indices of prices and quantities.

In fact, it has already been noted that no price and quantity levels are identifiable in the BLL format. Therefore, restrictions across equations should involve log changes. For example, it might be stipulated that the percent change in the aggregate price level is a weighted average of the percent changes in food and clothing prices:

$$wf(pf_r-pf_t) + wc(pc_r-pc_t) - (pa_r-pa_t) = 0 \qquad (6).$$

Here wf and wc are weights for food and clothing (wf, wc > 0, wf+wc = 1). Now equation (6) raises two further issues: For which periods (r,t) are the restrictions to apply ? And how are the weights (wf, wc) to be chosen ?

In the first place, the unrestricted BLL indices have been estimated with just $(T\text{-}1)^2$ degrees of freedom, so it is pointless to impose restrictions for all T(T-1)/2 price changes and an equal number of quantity changes ! To achieve parsimony and make allowance for sampling error, the restrictions might be imposed only for the first and last periods. Indeed, all T observations are represented in the term $pa_T - pa_1 = \Sigma(pa_t - pa_{t-1})$, where the summation runs from 2 to T. The interpretation would be that consistency in aggregation is to be expected on average, if not for each time interval:

$$wf(pf_T-pf_1) + wc(pc_T-pc_1) - (pa_T-pa_1) = 0 \qquad (7).$$

As for the choice of weights, the BLL approach is to derive the price and quantity indices symmetrically using all the data in the sample. Each price change is weighted by the quantities averaged over all periods, and each quantity change is weighted by the prices averaged over all periods. There is no need to choose an arbitrary "base period" which is perhaps not even in the sample. Unfortunately, this BLL principle is not easily extended to wf and wc in equation (7). For if these cross-equation weights are to be estimated jointly with the price changes, the problem becomes nonlinear. Then the computational simplicity and the hypothesis tests of the general linear model must be relinquished.

For practical purposes, therefore, the across-equation weights will have to be "prior information." If wf and wc are the value shares of food and clothing in some reference period, the restriction (7) is linear after all. Since the three equations (for f, c and a) have identical independent variables (just the ANOVA dummies), it is of no advantage to use seemingly-unrelated regression across equations. In fact, the usual restricted least-squares estimator for a single equation is applicable [e.g. Johnston (1984), chapter 6]. Moreover, the ANOVA structure permits further simplifications:

1. Using equations (2), obtain the three sets of unrestricted BLL estimates.
2. It turns out that restriction (7) changes only the estimates for periods 1 and T; the estimates for all other periods are unaffected.
3. For food, the adjusted log price index numbers are:

$$pf_1 + \lambda_p wf / T \quad \text{and} \quad pf_T - \lambda_p wf / T \tag{8}$$

where the Lagrange multiplier for restriction (7) is

$$\lambda_p = \frac{T[\, wf(pf_T-pf_1) + wc(pc_T-pc_1) - (pa_T-pa_1) \,]}{2(wf^2 + wc^2 + 1)} \tag{9}$$

4. For clothing, equation (8) applies with f replaced by c. For the aggregate index, equation (8) applies with f replaced by a and with wa = -1.

The quantity index also requires a restriction like (7):

$$wf(qf_T-qf_1) + wc(qc_T-qc_1) - (qa_T-qa_1) = 0 \quad . \tag{7'}$$

This leads to an adjustment factor λ_q, a Lagrange multiplier analogous to expressions (8) and (9), with qf, qc, and qa instead of pf, pc and pa. On the null hypothesis that the BLL index numbers aggregate consistently, the test statistic F $= (\lambda_p^2 + \lambda_q^2)(wf^2 + wc^2 + 1)/Ts^2$ has an F distribution with 2 and $(T\text{-}1)^2$ degrees of freedom. The hypothesis of consistent aggregation is rejected if F exceeds the tabular value at a specific level of significance.

Equations (7), (7'), (8), and (9) may be extended in an obvious way to handle aggregation over more than two commodity groups.

Multidimensional Indices

Although the discussion has focused on the analysis of prices and quantities, Banerjee (1977) showed how to generalize the ANOVA model to handle additional factors. As an illustration involving spatial comparisons, suppose that data are available on K commodities in T states of the USA. During a given year, every state has recorded, for each good, the volume sold, the ad valorem rate of sales tax, and the tax revenue collected. Summation over commodities then leads to the log-linear model:

$$v_{rts} = z + p_r + q_t + h_s + e_{rts} , \qquad (10)$$

where $\exp(v_{rts})$ is the aggregate revenue that would be obtained if the tax rates of state s were applied to the quantities sold in state t at the prices prevailing in state r. The three-way ANOVA would estimate interstate differences in price levels, sales volume, and tax rates. Compared to state s, state t might have higher tax rates on some items and lower rates on others. The estimate of h_s-h_t would be the basis for testing whether, on average, the tax rates in the two states are equal. Graybill [(1976), ch. 15] discusses inference in the three-way ANOVA design.

Nonspherical Errors

The OLS solution [equations (2)] is based on spherical errors, e_{rt}. However, the sample may be contaminated by either heteroscedasticity or autocorrelation. In principle, efficient estimates are then achieved by the generalized least-squares (GLS) transformation. In practice, the nonspherical pattern must be inferred from the OLS residuals. For finite samples, this "feasible" GLS estimator is not guaranteed to improve on OLS and could be much worse if the researcher's hunch about the error pattern is mistaken. Accordingly, the examination of residuals is best undertaken in a "what if" spirit, not as a formal exercise in statistical inference.

The BLL data, v_{rt}, are doubly dimensioned, perhaps as a time series, perhaps as a cross section of states or industries. This unusual structure should be reflected in conjectures about the error pattern. For example, one might specify that

$Var(e_{rt})$ = variance because prices aren't in lockstep

+ variance because quantities aren't in lockstep

+ covariance because prices and quantities aren't in lockstep.

Suppose that the first two error components are constant for all r and t, while the covariance depends on the proximity of the prices and quantities. Specifically,

$$Var(e_{rt}) = b_0 + b_1 c^d \qquad (11)$$

where $b_0 > 0$, $0 < c < 1$ and $d = |r-t|$. Equation (11) covers several cases:

- o If $b_1 = 0$ or $c = 0$, the errors are homoscedastic and OLS applies.
- o If $b_1 < 0$ while $b_0 > -b_1$, the contemporaneous data ($r = t$) are the most reliable, and the variance increases at longer lags.
- o If $b_1 > 0$, the contemporaneous data are the least reliable.

Equation (11) can be implemented by regressing e_{rt}^2 on c^d for several values of c and choosing the value that fits best. Then equations (2) are applied to the transformed observations $v_{rt}/(b_0+b_1c^d)$. (One might think that the denominator should be under a square-root sign. It would be if the ANOVA design matrix were also transformed to be used explicitly in a regression program. However, no such heavy computations are required.)

Exports of Raw Materials, 1976-1985

This section updates a previous study [Blankmeyer (1990)] in which the BLL method was applied to the commodity exports of developing nations. For the 33 raw materials listed in Table 1, the World Bank (1988) reports the quantities exported and the corresponding dollar prices (unit values). These annual series span 1976-1985, when commodity prices rebounded from a slump but then plummeted to record low levels in the recession of 1981-1983. In terms of equations (2), T = 10, so the sample includes $T^2 = 100$ observations.

With the minor exception of manganese ore, the items in Table 1 comprise the World Bank's value-weighted (VW) index. Each commodity's weight is its average value share in 1979-1981. The World Bank (1988, Table 21) deflates the VW price index by the cost of manufactures to obtain a terms-of-trade series. In the present paper, however, the undeflated VW price and quantity indices are compared to the respective BLL indices.

TABLE 1
Principal Commodity Exports of Developing Nations

Food and Beverages	Non-food Agriculture
1. cocoa (3.6)	19. cotton (4.6)
2. coffee (14.3)	20. jute (0.3)
3. tea (2.2)	21. rubber (5.4)
4. grain sorghum (0.8)	22. timber (8.1)
5. maize (2.5)	23. tobacco (2.7)
6. rice (3.8)	
7. wheat and meslin (1.3)	Metals and Minerals
8. sugar (12.3)	24. aluminum (1.4)
9. beef (2.0)	25. bauxite (1.2)
10. bananas (1.5)	26. copper (7.9)
11. oranges, etc. (1.2)	27. nickel (1.4)
12. copra (0.2)	28. tin (3.6)
13. groundnuts (0.5)	29. lead (1.0)
14. soybeans (1.5)	30. zinc (0.8)
15. coconut oil (1.1)	31. iron ore (4.4)
16. groundnut oil (0.4)	32. manganese ore (0.5)
17. palm oil (2.0)	33. phosphate rock (1.9)
18. oilseed cake and meal (3.6)	

1979-81 average value shares (percentages) appear in parentheses.
Source: World Bank (1988), Tables 10 and 11.

The main results of applying the World Bank data to equations (2), (3), and (4) may be summarized as follows:

- o The average annual change in the BLL price index is 1.5 percent.
- o Coincidentally, the average annual change in the BLL quantity index is also 1.5 percent.
- o Each of these average annual changes is subject to an estimated standard deviation of 0.27 percent.
- o The coefficient of determination, R^2, exceeds 0.99, indicating strong support for the hypothesis of lockstep variation in prices and quantities.
- o To screen for "outliers" in the data, the robust version of the BLL model [Blankmeyer (1991)] was also computed. The resulting indices are very similar to the OLS estimates, so the sample seems to be free of extreme observations.
- o The test for nonspherical errors, mentioned in a previous section, leads in this case to a corner solution: c = 0. In other words, the error variance does not appear to change systematically with the lag |r-t|.

The last two conclusions are bolstered to some extent by a casual perusal of the v_{rt} data themselves; they are all of the same magnitude since the prices of every year have been applied to the quantities of every year.

The BLL and VW price indices are compared in Figure 1. They generally agree on the direction of change if not on the magnitude.Both indices capture the collapse of commodity prices during the recession of the early 1980s.

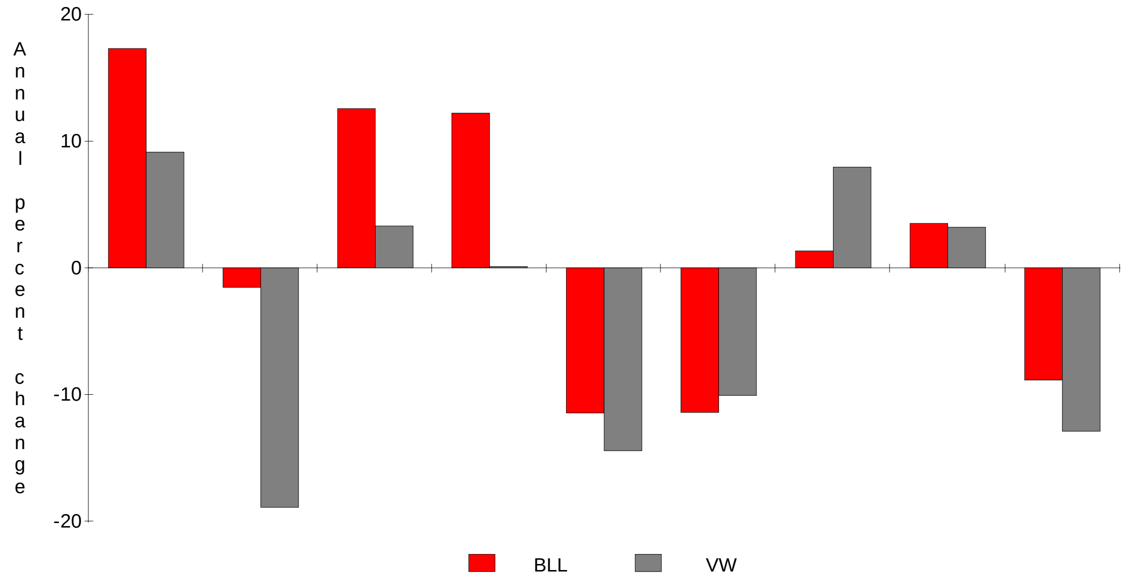
The World Bank data illustrate the use of restrictions to achieve consistency in aggregation. Table 1 displays three commodity groups and their respective weights --food and beverages (0.548), nonfood agriculture (0.211), and metals and minerals (0.241). Equations (2) were used to compute BLL indices for each group, and the log changes between 1976 (t=1) and 1985 (t=T=10) were evaluated. The Lagrange multipliers are λ_p = .0049 and λ_q = -0.0136, so F = 0.161 with 2 and 81 degrees of freedom. F is so small that one certainly cannot reject the hypothesis that these BLL price and quantity index numbers aggregate consistently.

"Statistical" and "Economic" Approaches

Theil (1960, p. 464) has mentioned "two distinct lines of approach to this subject....there is a statistical approach which is concerned with the specification of a central tendency of price and quantity ratios that is optimal in some statistical sense, and there is an economic approach which tries to specify price indices of varying batches of consumer goods corresponding with a constant level of satisfaction." _

Fig. 1 BLL and VW Price Indices

1977-1985



Like Theil's 1960 paper, the BLL approach is statistical; but a great deal of the recent literature on index numbers adopts the economic approach, which has been extended to include production theory as well as consumer theory. References include Theil (1965), W. E. Diewert (1976), D. W. Caves et al. (1982), and D. T. Slesnick (1991).

This modern economic approach to index numbers is based on the neoclassical theory of competitive markets. Rational consumers are supposed to maximize well-behaved utility functions subject to budget constraints, while price-taking firms apply efficient management to homothetic production processes. It is further assumed that the utility functions and production functions, while not directly observable, can be closely approximated by convex quadratic forms in the logarithms of the prices and quantities (translog functions). Finally, it is shown that the index numbers implied by this apparatus are of a certain type --the Theil-Tornqvist formulas, which are described in the references just cited.

However, the justifications for these formulas are in a sense contradictory. On the one hand, they are advocated because of the ingenious link to the neoclassical paradigm. On the other hand, the Theil-Tornqvist formulas are touted as "nonparametric": to compute them, one need not have estimated the substitution elasticities of the underlying utility functions or production functions. The sole ingredients are the same raw price and quantity data used by the BLL method and indeed by most other index-number formulas.

So while the neoclassical model cum translog approximation leads to the Theil-Tornqvist formula, the reverse is not true. In practice, one starts with price and quantity data. Plugging them into a nonparametric formula, one could never certify that the underlying utility functions or production functions are neoclassical. They are not identifiable in this context. The same point has been emphasized by R. L. Basmann et al. (1983, 1985); they have exhibited an interesting nonneoclassical utility function which is observationally indistinguishable from the neoclassical model.

To state the matter another way, imagine the hapless staff person in the Trade Ministry who must explain why the nation's export earnings collapsed last year. To what extent is the shortfall due to lower world prices ? To what extent is it due to weak domestic production ? Trying to sort out these effects, she decides to compute some price and quantity indices. Is it sensible to recommend that she choose a formula on the basis of its link --empirically unverifiable-- to the hypothetical constructs of neoclassical economics ?

The statistical approach to index numbers cannot be dismissed as "data without theory." The BLL method rests on an hypothesis that is simple but testable: apart from sampling error, prices change in lockstep; and so do quantities. As Pindyck and Rotemberg (1990, p. 1173) remark, "...this co-movement of prices applies to a broad set of commodities that are largely unrelated, i.e. for which the cross-price elasticities of demand and supply are close to zero. Furthermore, the co-movement is well in excess of anything that can be explained by the common effects of inflation, or changes in aggregate demand, interest rates, and exchange rates."

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