On Bartlett and Bartlett–Type Corrections

Francisco Cribari–Neto
Department of Economics, Southern Illinois University, Carbondale IL, 62901-4515, USA

and

Gauss M. Cordeiro
Departamento de Estatística, Universidade Federal de Pernambuco Cidade Universitária, Recife PE, 50740-540, Brazil

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Abstract

This paper reviews the literature on Bartlett and Bartlett-type corrections. It focuses on the corrections to the likelihood ratio, score and Wald test statistics. Three different Bartlett-type corrections which are equivalent to order n^{-1}, n being the sample size, are compared through simulation. One of the forms displayed superior behavior both in terms of size and power. We also use Monte Carlo simulation to examine the effect of independent variables and the impact of the number of nuisance parameters on the finite-sample behavior of some asymptotic econometric criteria in regression models.

1. Introduction

'Large sample' tests are commonly used in the applied work in econometrics and statistics since exact tests are not always available. These tests rely on what is called 'first order asymptotics', *i.e.*, they employ critical values obtained from a known limiting distribution. A natural question is then: Is this first order approximation a good approximation to the null distribution of the test statistic in use? A related question is: Can we do better? This paper will address these questions. We shall restrict ourselves to three large sample tests: the log-likelihood ratio (LR), score (S) and Wald (W) tests, since they are the most commonly used large sample tests. As is well known, these three statistics are asymptotically distributed as χ^2_q when the null hypothesis is true, where q is the number of restrictions imposed by H_0. However, it is also well known that this first order approximation may not work well in finite samples leading to size distortions. This paper addresses the issue of evaluating such an approximation and designing more accurate tests.

The question 'Can we do better?' can be approached from two distinct viewpoints. First, we can obtain a new distribution which is 'closer' to the true null distribution of our test statistic than the first order limiting distribution. Second, we can obtain a new test statistic which is better approximated by this first order limiting distribution. The focus of this survey will be on the latter since, unlike the former, there is no survey article or book with a detailed account of this literature. Readers interested in the first approach are referred to Barndorff-Nielsen and Cox (1979, 1989), Hall (1992), Kallenberg (1993), Pfanzagl (1980), Reid (1988, 1991), Rothenberg (1984) and the references therein. The purpose of our paper is to provide a unified review of the literature on Bartlett and Bartlett-type corrections, *i.e.*, corrections to be applied to test statistics and not to critical values. An issue of interest is how to define Bartlett-type corrections since it is possible to write the correction in different ways which are equivalent to a certain order of magnitude. We address this issue by Monte Carlo simulation. We also include a section that focuses on regression models, since these models are of central importance in the applied econometrics literature. We use the linear regression framework to address two important issues through simulation: the influence of the values of independent variables and the effect of nuisance parameters on the first order asymptotic approximation to some chi-squared econometric criteria. One of the simulations in this section involves the Breusch-Pagan test for heteroskedasticity which is commonly used in empirical applications that deal with cross-sectional data sets. Although Bartlett corrections constitute an important topic of research amongst statisticians, they have not found their appropriate space and usage in the econometrics literature, where size-corrections are almost always based on transformations of critical values obtained from Edgeworth expansions. We hope this survey will help narrow this gap.

2. Bartlett Correction to the Log–Likelihood Ratio Statistic

Generally speaking, the main difficulty of testing a null hypothesis using the log-likelihood ratio criterion lie not so much in deriving its closed-form expression—when it has one—but in finding its exact distribution, or at least a good approximation, when the null hypothesis is true. In an eventually influential paper, Bartlett (1937) proposed an improved LR statistic. His argument goes as follows. Suppose that under the null hypothesis $\mathrm{E}(LR) = q\{1 + b/n + O(n^{-2})\}$, where b is a constant that can be consistently estimated under H_0, n is the sample size and q is the difference between the dimensions of the parameter spaces under the alternative and null hypotheses. Then, the expected value of the transformed statistic $LR^* = LR/(1 + b/n)$ is closer to the one from a χ^2_q distribution than the expected value of LR. This became widely known as the *Bartlett correction*. He showed that for the test of homogeneity of variances the first three cumulants of LR^* agree with those of a χ^2_q distribution with error of order $n^{-3/2}$, thus providing strong grounds for one to think that the density of LR^* is better approximated by the asymptotic chi-squared distribution than is that of LR.

Bartlett (1938, 1947, 1954) obtained a number of adjustment factors in the area of multivariate analysis, and these factors became widely used for improving the large-sample chi-squared approximation to the null distribution of LR. Box (1949) used Bartlett's (1937) results to investigate in detail the general expression for the moments of the log-likelihood ratio statistic in the following cases: the test of constancy of variance and covariance of k sets of p-variate samples and Wilk's test for the independence of k sets of residuals, the ith set having p_i variates. He has shown, at least for these cases, that the modified statistic LR^* follows a χ^2_q distribution more closely than does the unmodified statistic LR. Box's results are applicable to all tests for which the Laplace transform of the test statistic can be explicitly written in terms of gamma functions and reciprocal gamma functions. In particular, it is possible to use these results to obtain $\mathrm{E}(LR)$ and $\mathrm{var}(LR)$. However, the results in Lawley (1956), McCullagh and Cox (1986) and Cordeiro (1993a) are better suited for econometric applications.

For regular problems, Lawley (1956) obtained expressions for the moments of certain derivatives of the log-likelihood function, and, via an exceedingly complicated derivation, gave a general formula for the null expected value of the log-likelihood criterion and showed that all cumulants of the Bartlett-corrected statistic for testing a composite hypothesis agree with those of the reference chi-squared distribution with error of order $n^{-3/2}$. A related reference is Beale (1960), who obtained an approximation to the asymptotic distribution of the residual sum of squares in a nonlinear normal regression model and gave an interpretation of the correction factor in terms of the curvature of a surface. Beale's paper has three contributions: it defines a measure of the intrinsic nonlinearity of a regression model as a function of the covariates and of the parameter values, shows how to get improved confidence regions for the parameter values of the model numerically, and shows how to choose a suitable transformation of the parameters that delivers near-linearity in the neighborhood of the maximum likelihood estimates. His results, however, are limited to normal models. In terms of the Bartlett correction, its main contribution was to give a geometric interpretation of the correction for normal models. This interpretation was later generalized to nonnormal models by McCullagh and Cox (1986).

Several correction factors applied to Markov chains were obtained by Sharp (1975) who has used Lawley's result to derive corrections for the test of the following hypotheses: that the transition probabilities are stable over time, that the chain is of a given order, and that several samples come from the same chain. Sharp's results cover most of the tests on Markov parameters used in practice. Williams (1976) derived Bartlett correction factors for log-linear models in complete multidimensional tables with closed-form estimators by expanding the criterion in Taylor series.

A further step on the improvement of the log-likelihood ratio statistic was taken by Hayakawa (1977), who obtained an asymptotic expansion of the null distribution of the log-likelihood ratio statistic LR for testing a composite null hypothesis H_0 against a composite alternative hypothesis H. He has shown that

$$\begin{aligned}\Pr[LR \leq z] =& F_q(z) + (24n)^{-1}[A_2 F_{q+4}(z) - (2A_2 - A_1)F_{q+2}(z) + (A_2 \\ &- A_1)F_q(z)] + o(n^{-1}),\end{aligned} \tag{1}$$

where $F_s(\cdot)$ is the cumulative distribution function of a chi-squared random variable with s degrees of freedom. A_1 and A_2 are functions of some cumulants of certain derivatives of the log-likelihood function. The error in (1) is $O(n^{-2})$ and not $O(n^{-3/2})$ as it is usually reported; see Barndorff-Nielsen and Hall (1988) for a proof of this result. However, since the Bartlett correction factor is given by $\rho = 1 + (12nq)^{-1}A_1$, Hayakawa's and Lawley's results are in conflict, unless $A_2 = 0$ in (1). The answer to this puzzle came ten years later with papers by Harris (1986) and Cordeiro (1987); see Hayakawa (1987). Harris showed that A_2 should not be present in (1) whereas Cordeiro showed that A_2 is equal to zero. This puzzle was recently revisited by Chesher and Smith (1994). They present an example in which A_2 is different from zero, and show that after one corrects Hayakawa's original formula for A_2, a zero value is always obtained. The main contribution of equation (1) with $A_2 = 0$ is that it provides a relatively simple demonstration that $LR^* = LR/\rho$ has a χ^2_q distribution with error $O(n^{-2})$. The term A_1 is a function of expected values of the first four log-likelihood derivatives and of the first two derivatives of these expected values with respect to the parameters of the model. The expression for A_1 holds for both simple and composite hypothesis, thus allowing for nuisance parameters. When nuisance parameters are present, A_1 can be calculated as the difference between two identical functions evaluated under the null and alternative hypotheses, respectively. This expression is general enough to be used in a number of econometric models since it is usually obtained from likelihood functions that obey the general regularity conditions stated in Cox and Hinkley (1974, Chapter 9), thus allowing one to handle independent, but not necessarily identically distributed observations. The applicability of the general expression for A_1 to regression models with serial correlation requires further restrictions on the cumulants of log-likelihood derivatives. The main problem of Lawley's formula is its interpretation since its individual terms are not parameter invariant. However, it can be widely used by econometricians when programmed in an algebraic manipulation language, such as Mathematica (Wolfram, 1991).

In recent years there has been a renewed interest in Bartlett corrections. Cordeiro (1983, 1987) derived closed-form expressions for Bartlett correction factors in generalized linear models (Nelder and Wedderburn, 1972) and discussed improved likelihood ratio goodness-of-fit tests. Williams' (1976) results are a special case of his results. Cordeiro (1995) presents extensive simulation results on the performance of a Bartlett-corrected deviance in generalized linear models focusing on gamma and log-linear models. Attfield (1995) focused on models that involve systems of equations, and derived Bartlett corrections to the log-likelihood ratio statistic in this case. A survey of various topics in regression analysis, including the asymptotic convergence of the adjusted residual sum of squares with a brief account of the role of the curvature, can be found in Johansen (1983). He considers the asymptotic distribution of the log-likelihood ratio statistic in normal nonlinear regression models and gives several theorems that deal with the convergence of LR and of the maximized log-likelihood, and interprets these results using Beale's (1960) measure and other measures of differential geometry. Johansen also gives a simple expression for the Bartlett correction in terms of Beale's curvature and an upper bound for the correction which is proportional to the minimal curvature of the model. Johansen's expression for the Bartlett correction was later generalized to nonlinear exponential family models by Cordeiro and Paula (1989).

An important non-regression case is that of one-parameter exponential family models. A simple, closed-form Bartlett correction for testing the null hypothesis that the parameter that indexes such models equals a given scalar was obtained by Cordeiro, Cribari–Neto, Aubin and Ferrari (1996). They then applied their result to 24 distributions in the exponential family, some of which are widely used in empirical applications in a variety of fields. A Bartlett correction for the natural exponential family had been previously given by McCullagh and Cox (1986).

Barndorff-Nielsen and Cox (1984a) gave an indirect method for computing Bartlett corrections under rather general parametric models by establishing a simple connection between the correction factor b and the norming constants of the general expression for the conditional distribution of a maximum likelihood estimator, namely

$$b = \left(\frac{A_0}{A}\right)^q \frac{n}{2\pi},$$

where A and A_0 are the norming constants of the general formula for the density of a maximum likelihood estimator conditional on an exact or approximate ancillary statistic (Barndorff-Nielsen, 1983) when this formula is applied to the full and null models, respectively. It is usually easier to obtain the Bartlett correction for special cases using Lawley's formula than using McCullagh and Cox's expression, since the former involves only moments of log-likelihood derivatives whereas the latter requires exact or approximate computation of the conditional distribution of the maximum likelihood estimates. When there are many nuisance parameters, it may not be easy to obtain ancillary statistics for these parameters, and hence the evaluation of McCullagh and Cox's formula can be quite cumbersome. The constants b, A and A_0 are usually functions of the maximal ancillary statistic, although to the relevant order of magnitude LR^* is independent of the ancillary statistic selected. They have also obtained various expressions for these quantities and, in particular, an approximation which does not require integration over the sample space for the one-parameter case. In another paper, Barndorff-Nielsen and Cox (1984b) considered the distribution of the log-likelihood ratio statistic for a number of types of censoring and sequential stopping rules related to Brownian motion, Poisson processes and survival analysis. It is clear from their examples that different stopping rules may lead to the same or to different Bartlett adjustments, that the stopping rule may be such that the asymptotic chi-squared distribution does not hold, and that there are intermediate cases in which the asymptotic chi-squared distribution holds but it is not Bartlett-correctable. Also, Cox (1984) considered the use of confidence intervals based on the adjusted log-likelihood ratio statistic and obtained adjustments for two special cases, one concerning two components of variance and the other being a slight generalization of the Behrens-Fisher problem.

An important question is whether the estimation of nuisance parameters alters the order of approximation after the Bartlett correction is applied. Cordeiro and Ferrari (1991) have shown that the unknown parameters in the correction of an asymptotic chi-squared test can be replaced by their restricted maximum likelihood estimates without changing the order of the approximation. A similar remark had been made by Lawley (1956) in the case of the log-likelihood statistic.

Correction factors to the signed version of the standardized log-likelihood ratio statistic $LR^{1/2}$ were derived by DiCiccio (1984) who showed through several examples that the signed log-likelihood ratio may be mean and variance adjusted, by means of cumulants, so as to approximate normality to order $n^{-1/2}$. In particular, he considered parameterizations which reduce the asymptotic bias and skewness of various pivotal quantities that arise in large-sample theory for models depending on an unknown scalar parameter.

Porteous (1985a) derived a correction factor for covariance selection models when the log-likelihood ratio statistic has a closed-form solution and illustrated the practical use of this correction through simulations. Also, Porteous (1985b) showed that the results of Cordeiro (1983) and Williams (1976) are equivalent for the test of nested decomposable log-linear regression models. These models involve a Poisson distributed dependent variable for which the logarithm of its mean is defined as a linear predictor that depends on unknown parameters and known independent variables.

Bartlett corrections for models defined by any one-parameter distribution in which the mean is a known function of a linear combination of unknown parameters were obtained by Cordeiro (1985), who generalized his own results of 1983. Further Bartlett adjustments for ten multivariate normal testing problems concerning structured covariance matrices from the simple connection between the adjustment factor and the norming constants of the conditional density of the maximum likelihood estimator were obtained by Møller (1986). In particular, Møller's results apply to real, complex and quaternion Wishart distributions and cover a number of tests. DiCiccio's (1986) paper is also related to the problem of computing correction factors for this general case.

When testing affine hypotheses in an exponential family, the "ideal" procedure is to calculate the exact similar test, or an approximation to it, based on the conditional distribution given the minimal sufficient statistic under the null hypothesis. Alternatively, there is a "primitive" approach in which the marginal distribution of a test statistic is used and any nuisance parameter appearing in the statistic is replaced by an estimate. Jensen (1986) showed that when using standardized log-likelihood ratio statistics, the "primitive" procedure is indeed an "ideal" procedure to order $n^{-3/2}$.

Since the log-likelihood ratio statistic is invariant under reparameterization, it is possible to express a large sample expansion of the test statistic and its expectation in terms of invariants. McCullagh and Cox (1986) used this fact to express the Bartlett adjustment factor in terms of invariant combinations of cumulants of the first two log-likelihood derivatives and gave it a geometric interpretation for some specific models. It should also be remarked that for the one-parameter model, the Bartlett correction can be easily interpreted in terms of the measures of the noninverse normality of the first derivative of the log-likelihood function and of Efron's (1975) curvature. The interpretation of the Bartlett correction in terms of this curvature in the multiparameter case was discussed by McCullagh and Cox (1986) in full generality, and by Ross (1987) for curved exponential families. Normal nonlinear regression models, which are of interest to econometricians, were discussed by McCullagh and Cox in terms of the curvature of the model. They give a simple expression for the Bartlett correction which coincides with Johansen's (1983) formula. It is also noteworthy that McCullagh and Cox's (1986) general formula coincides with Lawley's (1956) formula. The advantage of McCullagh and Cox's approach is its geometric interpretation, whereas the main advantage of Lawley's approach is that it can be more easily implemented to obtain the Bartlett correction for special cases.

Another method for obtaining Bartlett correction factors was described by

Barndorff-Nielsen and Blæsild (1986) which simplifies the numerical calculations in situations where one considers a number of hypotheses which are all linear in one and the same parameterization. This method relies on the cartesian tensorial nature of the cumulants of the log-likelihood derivatives and should be particularly convenient in connection with statistical packages of structure similar to GLIM (Generalized Linear Interactive Modeling). However, our experience is that it is usually more convenient to work with Lawley's expression.

A parameter-invariant form for the expected log-likelihood ratio criterion for statistical and econometric models consisting of a curved exponential family of distributions was presented by Ross (1987). His expression consists of two components: the first one measures the skewness and kurtosis associated with the tangent to the model at the null hypothesis whereas the second one reflects the nonplanarity of the model as a submanifold of the canonical parameter space. He also applied his expression to a general nonlinear regression model.

Improved log-likelihood ratio statistics for exponential family nonlinear models were obtained by Cordeiro and Paula (1989). They gave general closed-form expressions for Bartlett corrections in these models. Their expressions involve the general n^{-1} term in the null expected deviance for the class of generalized linear models and an unpleasant looking quantity which may be regarded as a measure of nonlinearity of the null expected deviance by the nonlinear parameters in the systematic component of the model.

Cordeiro (1993a) gave general matrix expressions for computing Bartlett corrections. Many recent papers have focused on deriving closed-form expressions for specific problems. For example, Moulton, Weissfeld and St. Laurent (1993) have obtained Bartlett corrections for logistic regressions, Cordeiro, Paula and Botter (1994) have derived corrections for the class of dispersion models proposed by Jørgensen (1987), Attfield (1991) and Cordeiro (1993b) have shown how to correct LR tests for heteroskedasticity, Wong (1991) has obtained a Bartlett correction factor for testing several slopes in regression models whose independent variables are subject to error, Wang (1994) derived the correction factor for testing the equality of normal variances against an increasing alternative, and Chesher and Smith (1993) have obtained Bartlett corrections for LR specification tests. A correction to the log-likelihood ratio statistic in regression models with Student-t errors was obtained by Ferrari and Arellano (1993), and similar corrections to heteroskedastic linear models and multivariate regression were obtained by Cribari–Neto and Ferrari (1995a) and Cribari–Neto and Zarkos (1995), respectively. An algorithm for computing Bartlett corrections was given by Jensen (1993); see also Andrews and Stafford (1993) and Stafford and Andrews (1993).

Bickel and Ghosh (1990) showed that it is possible to apply Bartlett corrections to improve Bayesian inference by showing that the posterior distribution of the log-likelihood statistic agrees with the chi-squared reference distribution with error of order $O(n^{-1})$ and that the posterior distribution of its Bartlett-corrected version is chi-squared when terms of order $O(n^{-2})$ and smaller are neglected. Ghosh and Mukerjee (1991, 1992) derived closed-form expressions for the Bartlett correction factor for Bayesian inference for: *(i)* $q = 1$ and no nuisance parameters, and *(ii)* $q = 1$ and one nuisance parameter when both parameters are orthogonal. A general formula for the Bartlett correction factor in this framework was obtained by DiCiccio and Stern (1993). Also, it was shown by DiCiccio and Stern (1994) that the errors in the χ^2 approximation to the sampling and posterior distributions of the adjusted log-profile-likelihood statistic can be reduced to order $O(n^{-2})$ by a Bartlett correction.

It is also possible to use Bartlett corrections in some nonparametric cases. For example, DiCiccio, Hall and Romano (1991) have shown that empirical likelihood (Owen, 1988, 1990) is Bartlett-correctable, and Chen and Hall (1993) extended this result to cover smoothed empirical likelihood.

Finally, it should be remarked that there is no guarantee that the corrected LR statistic for *discrete* data will yield an improvement in the asymptotic error rate of the chi-squared approximation. Indeed, Frydenberg and Jensen (1989) have shown by extensive numerical calculations that the Bartlett correction does not always deliver an error of order $O(n^{-2})$ in the lattice case.

3. Bartlett–Type Correction to the Score Statistic

The problem of developing a correction similar to the Bartlett correction to other test statistics was posed by Cox (1988) and solved three years later in full generality by Cordeiro and Ferrari (1991), and by Chandra and Mukerjee (1991) and Taniguchi (1991) for certain special cases; see also Mukerjee (1992). We shall focus on Cordeiro and Ferrari's results since they are more general in the sense that they allow for nuisance parameters.

An asymptotic expansion to the null distribution of the score statistic S was given by Harris (1985) as

$$\begin{aligned}\Pr[S \le z] =& F_q(z) + (24n)^{-1}[A_3 F_{q+6}(z) + (A_2 - 3A_3)F_{q+4}(z) + (3A_3 - 2A_2 \\ &+ A_1)F_{q+2}(z) + (A_2 - A_1 + A_3)F_q(z)] + o(n^{-1}), \qquad (2)\end{aligned}$$

where A_1, A_2 and A_3 are functions of some cumulants of log-likelihood derivatives. Also, he has shown that the first three cumulants of the score statistic are given by

$$\kappa_1(S) = q + \frac{A_1}{12n} + o(n^{-1}),$$
$$\kappa_2(S) = 2q + \frac{A_1 + A_2}{3n} + o(n^{-1}),$$
$$\kappa_3(S) = 8q + \frac{2(A_1 + 2A_2 + A_3)}{n} + o(n^{-1}).$$

As is well known, $\kappa_1(\chi^2_q) = q$, $\kappa_2(\chi^2_q) = 2q$ and $\kappa_3(\chi^2_q) = 8q$, and hence if we know A_1, A_2 and A_3 we can use the expressions above to find the first three cumulants of the score statistic to order n^{-1} and compare them with the cumulants of a χ^2_q random variable which is the basis for our first order approximation. Equation (2) holds for both simple and composite hypotheses. More importantly, this result implies that there exists no *scalar* transformation based on the test statistic which corrects all cumulants to a certain order of precision, as it is the case with the Bartlett correction to the LR statistic. All Harris' results enable us to do is to apply Hill and Davis' (1968) inverse formula to (2) in order to obtain transformed critical values to be used in the score test (Harris, 1985, p.657). The A's can be used to obtain corrections for models based on independent, but not necessarily identically distributed observations, thus covering a number of linear and nonlinear regression models (see Section 6).

A correction to be directly applied to the test statistic itself was obtained by Cordeiro and Ferrari (1991). They have shown that

$$S^* = S\left\{1 - \frac{1}{n}\sum_{j=1}^{3}\gamma_j S^{j-1}\right\}, \tag{3}$$

where $\gamma_1 = (A_1 - A_2 + A_3)/(12q)$, $\gamma_2 = (A_2 - 2A_3)/\{12q(q+2)\}$ and $\gamma_3 = A_3/\{12q(q+2)(q+4)\}$, is distributed as χ^2_q when terms of order smaller than n^{-1} are neglected. When the A's involve unknown parameters they should be replaced by their maximum likelihood estimates under H_0 and this does not affect the order of approximation of the correction. Note that the correction factor in (3) is a function of the unmodified statistic, and hence this correction is not a 'Bartlett correction' in the classical sense. Given its similarity with the Bartlett correction, however, it is termed *Bartlett-type correction*.

Cordeiro and Ferrari (1991) have also obtained a more general result which can be described as follows. Let T be a test statistic which is asymptotically distributed as χ^2_q. Then, it has been shown by Chandra (1985) that, under mild regularity conditions, it is possible to write

$$\Pr[T \leq z] = F_q(z) + \frac{1}{n}\sum_{i=0}^{k} a_i\, F_{q+2i}(z) \tag{4}$$

when terms of order $O(n^{-2})$ or smaller are neglected. Note that (4) implies that the distribution function to $O(n^{-1})$ of a test statistic asymptotically distributed as chi-squared is, under certain conditions, a linear combination of chi-squareds with degrees of freedom $q, q+2, \ldots, q+2k$. The a's are linear functions of the A's. For S, $k = 3$; for LR, $k = 1$. Cordeiro and Ferrari's (1991) result can then be stated as follows. Let $\mu'_i = 2^i\Gamma(i + q/2)/\{\Gamma(q/2)\}$, where $\Gamma(\cdot)$ is the gamma function, be the ith moment about zero of the χ^2_q distribution. Then, the modified test statistic

$$T^* = T\left\{1 - 2\sum_{i=1}^{k}(\sum_{j=i}^{k} a_j)(\mu'_i)^{-1}T^{i-1}\right\}$$

is distributed as χ^2_q to $O(n^{-1})$. This is a very general result which can be used to improve many important tests in econometrics and statistics. An extension of this result to Bartlett-type adjustments of order higher than a second order of approximation was recently proposed by Kakizawa (1994).

Building upon the result described above, Cordeiro, Ferrari and Paula (1993) and Cribari–Neto and Ferrari (1995b) obtained Bartlett-type corrections to score tests in generalized linear models for the cases of known and unknown dispersion, respectively. Bartlett-corrected score tests for heteroskedastic linear models are given in Cribari–Neto and Ferrari (1995a). Similar corrections for score tests in multivariate regression models were obtained by Cribari–Neto and Zarkos (1995), who have also shown through simulation that the Bartlett-type correction is more effective than the Cornish-Fisher correction of critical values in bringing the estimated sizes closer to their nominal levels. Bartlett-type corrections to score tests for heteroskedasticity were obtained by Cribari–Neto and Ferrari (1995c). Ferrari and Arellano (1993) derived improved score statistics for regression models with Student-t errors. General matrix formulae for the A's were given by Ferrari and Cordeiro (1994).

4. Bartlett–Type Correction to the Wald Statistic

The Wald test is very convenient to test nonlinear restrictions in linear models since it does not require estimation of the null model and therefore avoids nonlinear estimation. However, it has been shown by Gregory and Veall (1985), Lafontaine and White (1986) and others that a major drawback of this test is that it is not invariant to alternatively equivalent forms of the null hypothesis. Since many hypotheses of interest in economics are nonlinear (*e.g.*, restrictions implied by rational expectations models), it is important to develop corrections that can be reliably applied in finite-samples. Phillips and Park (1988) obtained an Edgeworth expansion to the null distribution of the Wald test of nonlinear restrictions (for simple hypotheses, an asymptotic expansion had been given by Hayakawa and Puri, 1985). Following Ferrari and Cribari–Neto (1993), their expansion can be written as

$$\begin{aligned}\Pr[W \le z] =& F_q(z) + \frac{1}{n}[a_3F_{q+6}(z) + a_2F_{q+4}(z) + a_1F_{q+2}(z) + a_0F_q(z) \\ &+ b_0f_q(z)] + o(n^{-1}),\end{aligned} \tag{5}$$

where $f_s(\cdot)$ is the probability density function of a chi-squared random variable with s degrees of freedom. Note that Phillips and Park's expansion is not in agreement with Chandra's result in (4), since (5) involves an extra term, $b_0f_q(\cdot)$, but b_0 equals zero in all examples considered by Phillips and Park (1988) and hence this extra term vanishes in such cases. However, it would be important to establish whether b_0 is *always* equal to zero. If this is the case, their expansion would no longer be in conflict with Chandra's (1985) result. It should be remarked that Phillips and Park (1988) obtained the expansion in (5) assuming that the limiting covariance matrix of the standardized estimator of the parameter vector is the identity matrix. Another interesting topic for further research is to obtain a similar expansion without making such an assumption. To this end, one would have to follow up on the results in the Appendix of their paper.

A Bartlett-type correction to the Wald test of nonlinear restrictions was obtained by Ferrari and Cribari–Neto (1993). They have shown that

$$W^* = W\left\{1 - \frac{1}{n}\sum_{j=0}^{3} \alpha_j W^{j-1}\right\}$$

is distributed as chi-squared to order n^{-1}, *i.e.*, $\Pr[W^* \leq z] = \Pr[\chi^2_q \leq z] + o(n^{-1})$. The α's here are obtained by rewriting (5) as

$$\Pr[W \leq z] = F_q(z) - f_q(z)\frac{1}{n}\sum_{j=0}^{3} \alpha_j z^j + o(n^{-1}).$$

As an example, consider the model in Lafontaine and White (1986): $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ with $\varepsilon_i \sim \mathrm{NID}(0, \sigma^2)$. The null hypothesis under test is $\mathrm{H}_0 : \beta_1^k = 1$ against a two-sided alternative. As shown by Lafontaine and White (1986), the size of the Wald test is highly sensitive to the value of k. For the Bartlett-type correction of the Wald test of this null hypothesis, we have that $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = -(2/3)(k-1)(k-2)$ and $\alpha_3 = (1/4)(k-1)^2$; see Phillips and Park (1988) and Ferrari and Cribari–Neto (1993).

It is also possible to design Bartlett-type corrections for other Wald tests. For example, Cribari–Neto and Ferrari (1995a) obtained improved Wald tests for heteroskedastic linear models and Cribari–Neto and Zarkos (1995) derived similar corrections to be used in multivariate regression. An Edgeworth expansion for the nonnull distribution of W in generalized linear models was given by Cordeiro, Botter and Ferrari (1994), who have also compared the power of the Wald test under Pitman alternatives to the powers of the log-likelihood ratio and score tests.

5. Alternative Forms of the Bartlett–Type Correction

Bartlett-type corrections are usually defined as $T_1^* = T(1 - B/n)$, where B is a polynomial on the unmodified statistic, as in equation (3). However, there are alternative definitions of Bartlett-type corrections which are equivalent to order n^{-1}. For example, $T_2^* = T/(1 + B/n)$ and $T_3^* = T\exp\{-B/n\}$ are equivalent to T_1^* when terms of order smaller than n^{-1} are ignored. Note that the latter has the advantage of always delivering nonnegative corrected statistics. In this section we compare these three alternative forms through Monte Carlo simulation.

We consider three cases. Let Z_1, Z_2 and Z_3 be distributed as $\mathrm{N}(\theta_1, \theta_2), \mathrm{IG}(\theta_3, \theta_4)$ and $\mathrm{G}(\theta_5, \theta_6)$, respectively. That is, Z_1 is normally distributed with mean θ_1 and variance θ_2, Z_2 is distributed as inverse Gaussian with mean θ_3 and scale parameter θ_4, and Z_3 has a gamma distribution with mean θ_5 and scale parameter θ_6. [The inverse Gaussian distribution is also known as Wald's distribution or the first passage time distribution of a Brownian motion with positive drift.] Our interest is in testing $\mathrm{H}_0 : \theta_2 = \theta_2^{(0)}$, $\mathrm{H}_0 : \theta_4 = \theta_4^{(0)}$ and $\mathrm{H}_0 : \theta_6 = \theta_6^{(0)}$ against two-sided alternatives, assuming that the means are unknown. For the first two cases, $A_1 = -6$, $A_2 = 12$ and $A_3 = 40$, whereas for the latter case the A's are functions of $\theta_6^{(0)}$ and of the trigamma and tetragamma functions evaluated at this point. The size simulations were conducted setting $\theta_1 = 0$, $\theta_2 = 1$, $\theta_3 = 3$, $\theta_4 = 1$, $\theta_5 = 2$ and $\theta_6 = 0.5$, and the number of replications was fixed at 10,000. In order to generate random numbers from an inverse Gaussian distribution we used the algorithm in Devroye (1986, p.149); see also Michael, Schucany and Haas (1976) and Padgett (1978). Rejection rates under the null hypothesis for the normal, inverse Gaussian and gamma distributions are given in Tables 1, 2 and 3, respectively, for $n = 10, \ldots, 40$ and $\alpha = 10\%, 5\%, 1\%$. The standard errors for the estimated percentages of rejections corresponding to these nominal levels are 0.95%, 0.22% and 0.10%.

We also present results on the powers of tests. Here the data are generated under the alternative hypothesis using $\theta_2 = \theta_4 = 0.7$ and $\theta_6 = 0.4$. The results based on 10,000 replications are reported in Tables 4, 5 and 6 for $n = 10, \ldots, 40$ and $\alpha = 10\%, 5\%$. Finally, Figure 1 displays the size distortions and estimated powers of the score test (Score) and its three corrected versions (Bartlett1, Bartlett2 and Bartlett3) for $\alpha = 5\%$.

The figures in Tables 1, 2 and 3 show that the three Bartlett-corrected statistics have a similar size behavior, and that all corrected tests outperform the original score test, especially when the number of observations is small. In particular, S_2^* has a slightly superior behavior for small samples, followed by S_3^* and then S_1^*.

The power simulations were conducted using tabulated and not estimated critical values. This was done mainly because none of the tests is oversized. Power comparisons based on estimated critical values can be misleading since investigators do not use such critical values in empirical applications. They become necessary, however, when one of the tests is oversized, since in this case the test's ability to reject the null hypothesis comes mainly from its oversizedness, and not from its ability to detect when the this hypothesis is not true. For example, a test formulated as 'always reject' would have maximum power, but this would come from the fact that its size was also 100%. The level of significance α is such that probabilities of type I error *greater* than α are undesirable (Bickel and Doksum, 1977, p.168), and one usually focuses on tests with size less than or equal to α. Put differently, we are

TABLE 1: Testing the Variance of a Normal Distribution–Size

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	6.8	9.1	9.7	9.4
	5	3.0	4.2	4.8	4.4
	1	1.1	1.1	1.3	1.2
20	10	8.3	9.5	9.7	9.7
	5	3.6	4.5	4.8	4.7
	1	0.8	0.8	0.9	0.8
30	10	8.9	9.9	10.0	9.9
	5	3.9	4.6	4.7	4.6
	1	1.0	0.9	1.0	1.0
40	10	9.7	10.6	10.7	10.6
	5	4.5	5.0	5.0	5.0
	1	0.9	0.9	1.0	1.0

TABLE 2: Testing the Scale Parameter of an Inverse Gaussian Distribution–Size

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	7.0	8.8	9.4	9.1
	5	3.0	4.2	4.9	4.5
	1	1.1	0.9	1.3	1.0
20	10	8.9	9.9	10.1	10.0
	5	3.9	4.9	5.2	5.1
	1	1.0	1.0	1.2	1.0
30	10	8.6	9.4	9.5	9.5
	5	4.1	4.8	4.9	4.8
	1	0.8	0.8	0.9	0.8
40	10	9.5	10.1	10.2	10.2
	5	4.7	5.2	5.2	5.2
	1	1.3	1.4	1.4	1.4

TABLE 3: Testing the Scale Parameter of a Gamma Distribution–Size

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	7.3	9.1	9.6	9.3
	5	3.2	4.0	4.5	4.2
	1	1.0	0.9	1.1	1.0
20	10	8.4	9.3	9.5	9.4
	5	3.8	4.6	4.7	4.7
	1	0.8	0.7	0.8	0.8
30	10	9.2	9.7	9.8	9.8
	5	4.4	4.9	5.0	5.0
	1	1.0	1.0	1.1	1.1
40	10	9.8	10.3	10.4	10.3
	5	4.8	5.2	5.2	5.2
	1	1.0	1.0	1.0	1.0

TABLE 4: Testing the Variance of a Normal Distribution–Power

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	8.1	12.9	13.6	13.2
	5	0.7	3.5	4.1	3.8
20	10	20.1	23.4	23.6	23.5
	5	7.0	10.3	10.6	10.4
30	10	30.4	32.7	32.8	32.7
	5	15.1	17.7	17.9	17.8
40	10	38.6	40.5	40.6	40.5
	5	21.6	24.0	24.1	24.1

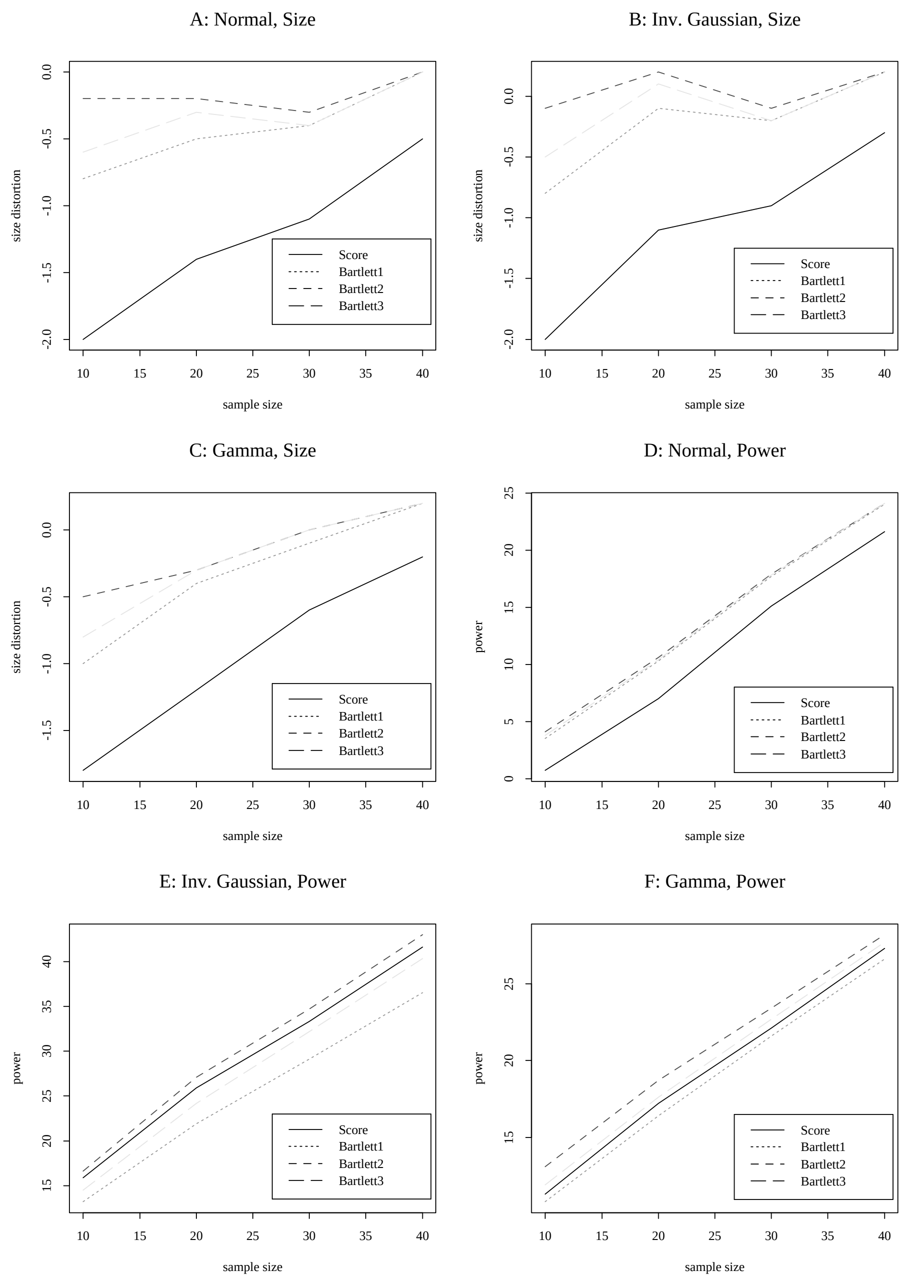
TABLE 5: Testing the Scale Parameter of an Inverse Gaussian Distribution–Power

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	21.1	18.6	22.6	19.9
	5	15.9	13.2	16.6	14.5
20	10	33.2	29.5	34.7	32.1
	5	25.9	21.9	27.1	24.2
30	10	41.6	37.5	42.8	40.7
	5	33.3	29.1	34.7	32.2
40	10	50.5	45.5	51.4	49.3
	5	41.6	36.5	43.0	40.3

TABLE 6: Testing the Scale Parameter of a Gamma Distribution–Power

n	α	$\Pr[S \geq z_\alpha]$	$\Pr[S_1^* \geq z_\alpha]$	$\Pr[S_2^* \geq z_\alpha]$	$\Pr[S_3^* \geq z_\alpha]$
10	10	16.5	16.3	18.4	17.4
	5	11.3	10.8	13.1	11.9
20	10	24.3	23.4	25.5	24.8
	5	17.2	16.4	18.7	17.6
30	10	30.2	29.4	31.0	30.5
	5	22.1	21.6	23.4	22.7
40	10	35.3	34.6	36.1	35.8
	5	27.3	26.6	28.2	27.7

Figure 1: Size Distortions and Estimated Powers



interested in comparing the powers of *level* α (as opposed to *size* α) tests.

It is clear from Tables 4, 5 and 6 and from Figure 1 that S_2^* has the best power performance. For the normal distribution, all three corrected tests had slightly higher power than the original test. The power behavior of the corrected tests was similar. When the data were generated from an inverse Gaussian distribution, S_2^* was the most powerful test statistic, followed by S, S_3^* and S_1^*. For the gamma distribution, S_2^* was followed by S_3^*, S and S_1^*. Although S_1^* is the most used version of the Bartlett-type corrected score statistic, the other alternative forms considered here were slightly more effective in reducing the size distortion of the test and also more powerful under the alternative hypothesis.

6. Regression

Most econometric applications involve regression models, and there are a number of Bartlett and Bartlett-type corrections that can be of some use. This section looks at some of them. It also sheds some light on the effect of covariate values and nuisance parameters on the convergence to the limiting null distribution of some test statistics using Monte Carlo simulation.

At the outset, consider the linear regression model $y = X\beta + \varepsilon$, where y, the dependent variable, and ε, the random disturbance, are n-vectors, X is an $n \times p$ matrix of covariates and β is a p-vector of unknown parameters. For each i, $i = 1, 2, \ldots, n$, $\varepsilon_i \sim \text{NID}(0, \sigma_i^2)$ with $\sigma_i^2 = h(w_i'\alpha)$, $w_i' = (1 \; v_i')$ is a $1 \times (q+1)$ vector of exogenous variables, α is a $(q+1)$-vector of parameters and $h(\cdot)$ is a positive valued function which does not depend on i. It is common practice to use Breusch and Pagan's (1979) score (Lagrange multiplier) statistic to test the null hypothesis of homoskedasticity ($\alpha_1 = \ldots = \alpha_q = 0$) against the alternative of heteroskedasticity of unknown form. A well known problem associated with this test is its tendency to under-reject the null hypothesis when heteroskedasticity is not present. Closed-form expressions for the A's for this test can be found in Honda (1988) and Cribari–Neto and Ferrari (1995c). In particular, Cribari–Neto and Ferrari (1995c) have shown that

$$A_1 = 24q(p-1) - 24n\,\text{tr}(H_d J_d) + 6n\mathbf{1}'J_d H J_d \mathbf{1} + 12n\mathbf{1}'(H * J * J)\mathbf{1},$$
$$A_2 = -24q(q+2) + 36n\,\text{tr}(H_d * H_d) - 24n\mathbf{1}'H_d H J_d \mathbf{1},$$
$$A_3 = 24n\mathbf{1}'H_d H H_d \mathbf{1} + 16n\mathbf{1}'H * H * H\mathbf{1},$$

where $J = X(X'X)^{-1}X'$, $H = V(V'V)^{-1}V'$, $V = (v_1 - \bar{v}, \ldots, v_n - \bar{v})'$, $J_d = \text{diag}\{j_{11}, \ldots, j_{nn}\}$, $H_d = \text{diag}\{h_{11}, \ldots, h_{nn}\}$, $\mathbf{1}$ is an n-vector of ones, and '$*$' denotes the Hadamard product. These formulae can be used to obtain numerical values for A_1, A_2 and A_3 in empirical applications or closed-form expressions for special models. Bartlett corrections for log-likelihood ratio tests for heteroskedasticity can be found in Attfield (1991) and Cordeiro (1993b).

Now consider a simple linear regression model given by $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, 2, \ldots, n$, $\varepsilon_i \sim \text{NID}(0, \sigma_i^2)$ where $\sigma_i^2 = h(\alpha_0 + \alpha_1 x_i)$, which is a special case of the heteroskedastic model introduced above. Cribari–Neto and Ferrari (1995c) have shown that for this simple regression model the expressions for the A's given above reduce to $A_1 = -6(8 + \gamma_{2x} - 3\gamma_{1x}^2)$, $A_2 = 12(3 + 3\gamma_{2x} - 2\gamma_{1x}^2)$ and $A_3 = 40\gamma_{1x}^2$, where γ_{1x} and γ_{2x} are the sample measures of skewness and excess kurtosis of the independent variable x. This then suggests that the covariate values can play an important role in the quality of the asymptotic chi-squared approximation that is used to perform the Breusch-Pagan test. To illustrate this point, we perform a Monte Carlo simulation experiment using $\beta_0 = \beta_1 = 1$, $n = 30$ and 10,000 replications. The data are generated under H_0 with $\sigma_i^2 = \sigma^2 = 1$. The values of x consist of an evenly spaced sequence of $n - 2 = 28$ points from -1 to 1 and endpoints $-a$ and a. It is clear that the sample excess kurtosis of x increases with a. Figure 2 shows the estimated size distortions of the Breusch-Pagan test and its Bartlett-corrected version (obtained in a similar fashion as what we called 'Bartlett 2' in the previous simulation experiment) for the nominal level of 5%. The results for other nominal levels were similar.

It is clear that the size performance of the Breusch-Pagan deteriorates as a (and consequently γ_{2x}) increases. The Bartlett-type correction is effective for moderate values of a (say, $a \leq 2.5$), delivering estimated sizes that are closer to the 5% nominal level. For large values of a, it tends to overcorrect the score statistic. These results show that in some cases the covariate values can affect the size performance of asymptotic tests considerably. The expressions for the A's should reveal which features of the model affect this performance (to order n^{-1}). For example, the expressions for the A's for the Bartlett-type correction of the Breusch-Pagan test reveal that the sample skewness and the sample excess kurtosis of the independent variable affect the first order approximation of the test.

Another important factor that can affect the first order approximation of asymptotic econometric criteria is the number of nuisance parameters. To illustrate this point, we consider a linear normal regression model. The A's for tests on the β vector were obtained by Cribari–Neto and Ferrari (1995) as $A_1 = 12q(p - q)$, $A_2 = -6q(q + 2)$ and $A_3 = 0$; see below. (It can be shown that these A's also hold for a large class of nonlinear normal regression models; see Ferrari, Uribe–Opazo and Cribari–Neto, 1995.) It is then clear that the number of nuisance parameters $p - q$, where q is the number of restrictions imposed by H_0, has an impact on A_1, and thus on the finite-sample performance of the score test. Such an impact can be made clear with the help of a simulation experiment. We consider ten models. The first is $y = \beta_9 x_9 + \beta_{10} x_{10} + \varepsilon$ $(p - q = 0)$, the second is $y = \beta_0 + \beta_9 x_9 + \beta_{10} x_{10}$ $(p - q = 1)$, the third is $y = \beta_0 + \beta_1 x_1 + \beta_9 x_9 + \beta_{10} x_{10} + \varepsilon$ $(p - q = 2)$, and so forth, until the last model, which is defined as $y = \beta_0 + \beta_1 x_1 + \ldots + \beta_8 x_8 + \beta_9 x_9 + \beta_{10} x_{10} + \varepsilon$ $(p - q = 9)$. The null hypothesis under test in all cases is $\text{H}_0 : \beta_9 = \beta_{10} = 0$. The number of observations was set at 30 and the number of replications at 10,000. All independent variables were chosen as random draws from a U(0, 1) distribution, the errors were obtained from a N(0, 1) distribution, and all nuisance parameters (if any) were set equal to 1. Figure 3 plots the estimated size distortions of the score test and its Bartlett-corrected version (again using the second specification, as defined in the previous section) for the nominal level of 5%. The results for other nominal levels were similar and are not reported.

It is clear that the number of nuisance parameters has a substantial impact on the conventional first order chi-squared approximation. The score test is slightly undersized when $p-q=0$, and becomes oversized as $p-q$ increases, being extremely oversized when $p-q$ becomes large. The Bartlett-corrected test holds its size close to the nominal 5% level remarkably well.

Simulation results for regression models involving gamma distributed random variables can be found in Cordeiro and Cribari–Neto (1993). Their simulation results also show that the Bartlett-type correction is effective in bringing the actual size of the score test closer to its nominal level.

A fairly general framework for working with regression models is the class of generalized linear models introduced by Nelder and Wedderburn (1972); see also the comprehensive book by McCullagh and Nelder (1989). These models can be briefly described as follows. $y=(y_1,\ldots,y_n)'$ is a vector of independent variables and each y_i has a probability or density function in the exponential family:

$$\pi(y;\theta_i,\phi)=\exp\{\phi[y\theta_i-b(\theta_i)+c(y)]+a(y,\phi)\}, \tag{6}$$

where $a(\cdot,\cdot)$, $b(\cdot)$ and $c(\cdot)$ are known functions and θ_i and ϕ are (possibly unknown) parameters. We have that $\mathrm{E}(y_i)=\mu_i=b'(\theta_i)$ and $\mathrm{var}(y_i)=\phi^{-1}V_i$, where ϕ^{-1} is the dispersion parameter, $V=V(\mu)=\mathrm{d}\mu/\mathrm{d}\theta$ is the variance function, and $\theta=\int(1/V)\mathrm{d}\mu=q(\mu)$ is a strictly monotonic function of the mean. The linear predictor is defined as $\eta=\sum_{j=1}^{p}\beta_j x_j=X\beta$, where X is an $n\times p$ matrix of covariates (of rank p) and β is a p-vector of unknown parameters. The mean of the dependent variable is then related to the linear predictor through a strictly monotonic, twice differentiable link function $d(\mu)=\eta$. The link function here is assumed known; for recent developments involving unknown link functions, see Mallick and Gelfand (1994) and Weisberg and Welsh (1994). Generalized linear models have normal linear regression, Poisson regression, gamma regression, inverse Gaussian regression, and logit and probit models as special cases. For example, in the normal linear regression model, $V=1$, ϕ^{-1} is the error variance and $\mu=\eta$.

Suppose we partition the β vector as $(\beta_1',\beta_2')'$, where $\beta_1=(\beta_1,\ldots,\beta_q)'$ $(q\le p)$ and $\beta_2=(\beta_{q+1},\ldots,\beta_p)'$, thus inducing a partition of the covariate matrix as $X=[X_1\ X_2]$, and we want to test the null hypothesis $\mathrm{H}_0:\beta_1=\beta_1^{(0)}$, where $\beta_1^{(0)}$ is a q-vector of known constants, against a two-sided alternative. The score statistic for this test is given by

$$S=\tilde{s}'\tilde{W}^{1/2}X_1(\tilde{R}'\tilde{W}\tilde{R})^{-1}X_1'\tilde{W}^{1/2}\tilde{s},$$

Figure 2: Heteroskedasticity Test

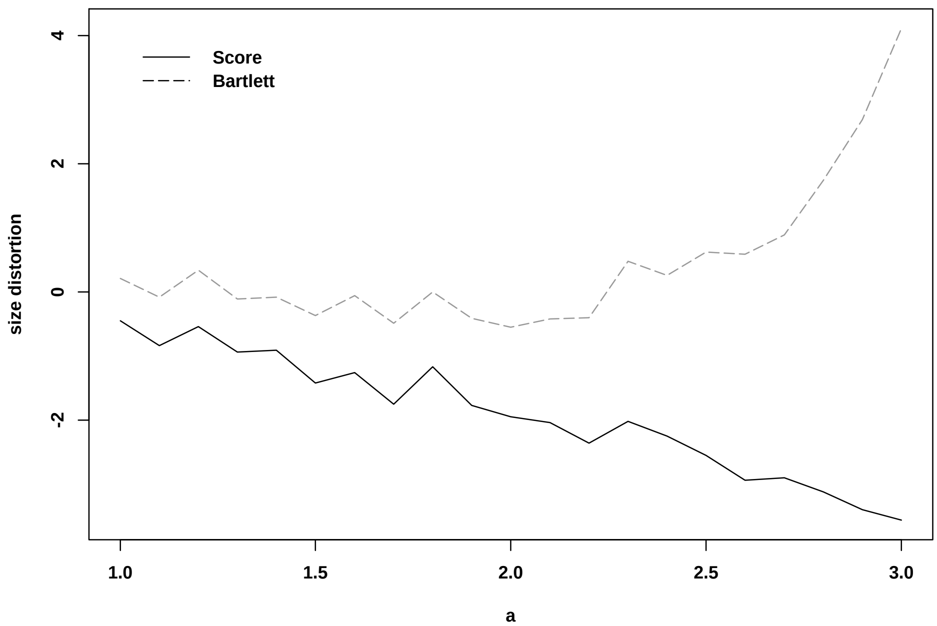
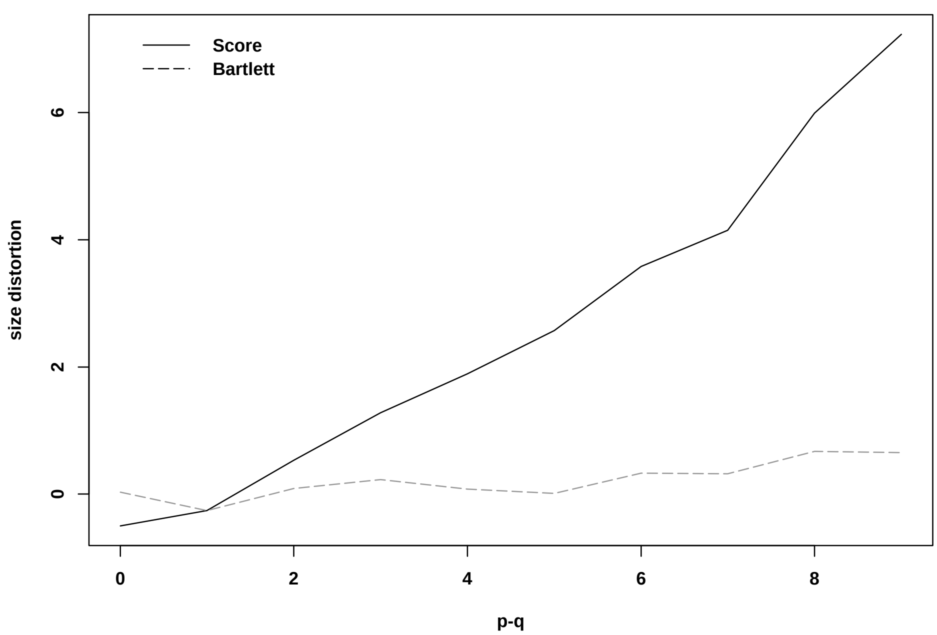


Figure 3: Exclusion of Variables



where $W = \mathrm{diag}\{w_1, \ldots, w_n\}$ with $w_i = (\mathrm{d}\mu_i/\mathrm{d}\eta_i)^2/V_i$, $s = (s_1, \ldots s_n)'$ with $s_i = \phi^{1/2}(y_i - \mu_i)/V_i^{1/2}$, $R = X_1 - X_2(X_2'WX_2)^{-1}X_2'WX_1$ and tildes denote evaluation at the restricted maximum likelihood estimates. When the dispersion parameter is unknown we have a two-parameter full exponential family with canonical parameters ϕ and $\phi\boldsymbol{\theta}$, and the term $a(y, \phi)$ in (6) can be written as $a(y, \phi) = d_1(\phi) + d_2(\phi)$. Different distributions yield different functions for $d_1(\phi)$ and $d_2(y)$. For example, for the normal distribution with variance ϕ^{-1}, $d_1(\phi) = \log(\phi/2\pi)/2$ and $d_2(y) = 0$. The A's that define the Bartlett-type correction to the score statistic can be written as (Cribari–Neto and Ferrari, 1995b) $A_1 = A_{1,\beta} + A_{1,\beta\phi}$, $A_2 = A_{2,\beta} + A_{2,\beta\phi}$ and $A_3 = A_{3,\beta} + A_{3,\beta\phi}$, where $A_{1,\beta}$, $A_{2,\beta}$ and $A_{3,\beta}$ are the A's for the known dispersion case, and $A_{1,\beta\phi}$, $A_{2,\beta\phi}$ and $A_{3,\beta\phi}$ are some extra terms that account for the uncertainty involved in the estimation of ϕ^{-1}, the dispersion parameter. We have that (Cordeiro, Ferrari and Paula, 1993)

$$\begin{aligned}A_{1,\beta} =& n\,\phi^{-1}\{3\,\mathbf{1}'FZ_{2d}(Z - Z_2)Z_{2d}F\,\mathbf{1} + 6\,\mathbf{1}'FZ_{2d}Z_2(Z - Z_2)_d(F - G)\,\mathbf{1}\\ &- 6\,\mathbf{1}'F\{Z_2^{(2)} * (Z - Z_2)\}(2G - F)\,\mathbf{1} - 6\,\mathbf{1}'H(Z - Z_2)_d Z_{2d}\,\mathbf{1}\},\end{aligned}$$

$$\begin{aligned}A_{2,\beta} =& n\,\phi^{-1}\{-3\,\mathbf{1}'(F - G)(Z - Z_2)_d Z_2(Z - Z_2)_d(F - G)\,\mathbf{1}\\ &- 6\,\mathbf{1}'FZ_{2d}(Z - Z_2)(Z - Z_2)_d(F - G)\,\mathbf{1}\\ &- 6\,\mathbf{1}'(F - G)\{(Z - Z_2)^{(2)} * Z_2\}(F - G)\,\mathbf{1} + 3\,\mathbf{1}'B(Z - Z_2)_d^{(2)}\,\mathbf{1}\}\end{aligned}$$

and

$$\begin{aligned}A_{3,\beta} =& n\,\phi^{-1}\{3\,\mathbf{1}'(F - G)(Z - Z_2)_d(Z - Z_2)(Z - Z_2)_d(F - G)\,\mathbf{1}\\ &+ 2\,\mathbf{1}'(F - G)(Z - Z_2)^{(3)}(F - G)\,\mathbf{1}\}.\end{aligned}$$

Here, $Z = X(X'WX)^{-1}X'$, $Z_2 = X_2(X_2'WX_2)^{-1}X_2'$, $Z_d = \mathrm{diag}\{z_{11}, \ldots, z_{nn}\}$, $Z_{2d} = \mathrm{diag}\{z_{211}, \ldots, z_{2nn}\}$, $F = \mathrm{diag}\{f_1, \ldots, f_n\}$, $G = \mathrm{diag}\{g_1, \ldots, g_n\}$, $B = \mathrm{diag}\{b_1, \ldots, b_n\}$, and $H = \mathrm{diag}\{h_1, \ldots, h_n\}$, with

$$f = \frac{1}{V}\frac{d\mu}{d\eta}\frac{d^2\mu}{d\eta^2}, \quad g = \frac{1}{V}\frac{d\mu}{d\eta}\frac{d^2\mu}{d\eta^2} - \frac{1}{V^2}\frac{dV}{d\mu}\left(\frac{d\mu}{d\eta}\right)^3,$$
$$b = \frac{1}{V^3}\left(\frac{d\mu}{d\eta}\right)^4\left\{\left(\frac{dV}{d\mu}\right)^2 + V\frac{d^2V}{d\mu^2}\right\}, \quad h = \frac{1}{V^2}\frac{dV}{d\mu}\left(\frac{d\mu}{d\eta}\right)^2\frac{d^2\mu}{d\eta^2} + \frac{1}{V^2}\frac{d^2V}{d\mu^2}\left(\frac{d\mu}{d\eta}\right)^4.$$

We also have that (Cribari–Neto and Ferrari, 1995b)

$$A_{1,\beta\phi} = \frac{6q\{d_{(3)} - (p - q - 2)d_{(2)}\}}{d_{(2)}^2}, \quad A_{2,\beta\phi} = \frac{3q(q + 2)}{d_{(2)}},$$

where $d_{(2)} = d_{(2)}(\phi) = \phi^2 d_1''(\phi)$ and $d_{(3)} = d_{(3)}(\phi) = \phi^3 d_1'''(\phi)$. Also, $A_{3,\beta\phi} = 0$.

The A's used in the simulations of normal linear models (Figure 3) were obtained as special cases of these A's for generalized linear models. It should also be noted that similar results for Poisson regression and logit and probit models that are commonly used in the econometrics literature can also be obtained as special cases of the formulae above. For Poisson models, $V = \mu$ and for logit and probit models $V = \mu(1-\mu)$. A generalization of the result presented above to nonlinear models can be found in Ferrari, Uribe–Opazo and Cribari–Neto (1995).

We can also consider the test of the null hypothesis $\mathrm{H}_0 : \phi = \phi^{(0)}$ against the alternative $\mathrm{H}_1 : \phi \neq \phi^{(0)}$, where $\phi^{(0)}$ is a given scalar. For example, in Poisson regression models one might want to test the hypothesis that $\phi = 1$ against the alternative of overdispersion or underdispersion. The A's for the Bartlett-type correction of the score statistic are (Cordeiro, Ferrari and Paula, 1993)

$$A_1 = -\frac{3p(p-2)}{d_{(2)}}, \quad A_2 = -\frac{3\{2pd_{(3)} + d_{(4)}\}}{d_{(2)}^2}, \quad A_3 = -\frac{5d_{(3)}^2}{d_{(2)}^3},$$

where $d_{(4)} = d_{(4)}(\phi) = \phi^4 d_1^{iv}(\phi)$.

Here we have focused on Bartlett-type corrections for score tests. Similar Bartlett corrections for log-likelihood ratio statistics in generalized linear models can be found in Cordeiro (1983, 1987).

7. Some Remarks on Power and Bias

Bartlett and Bartlett-type corrections are designed to bring the actual size of asymptotic tests close to their respective nominal sizes. In most cases, they are effective in doing so. They are not intended, however, to be corrections to increase the power of the test. It is important to bear in mind that these corrections can lead to a loss in power, much in the same way as the power of Durbin's h statistic (Durbin, 1970), a transformation of the traditional Durbin-Watson statistic, can be lower than the power of the Durbin-Watson test in regression models with lagged dependent variables; see Inder (1984, 1986, 1990). However, an important result is that the untransformed statistic and its Bartlett-corrected version have the same local power to order $n^{-1/2}$. This result follows from Theorem 1 in Cox and Reid (1987). More precisely, let T be a test statistic with null distribution χ_q^2, and T^* a Bartlett-corrected statistic obtained as a transformation of T. Then, under local (Pitman) alternatives, $\Pr[T^* \geq c] = \Pr[T \geq c] + o(n^{-1/2})$.

It should also be mentioned that a similar literature is that of bias correction, where the asymptotic bias is used to obtain estimators that are bias-free to order (say) n^{-1}, which is an appealing alternative to computer-intensive bias correction techniques, such as the one described by MacKinnon and Smith (1995). Cordeiro and McCullagh (1991) used this approach to obtain bias-corrected maximum likelihood estimators for the class of generalized linear models, and Cordeiro and Klein (1994) to obtain similar bias corrections for maximum likelihood estimators in ARMA models. Simulation results are given in Cordeiro and Cribari–Neto (1993). In particular, it is possible to obtain bias-corrected estimators for a number of regression models as a special case of Cordeiro and McCullagh's (1991) results. Both Cordeiro and McCullagh's and Cordeiro and Klein's results are based on the general formulae given by McCullagh (1987). For an alternative approach, see Cadigan (1994).

8. Concluding Remarks

Bartlett-type corrections constitute a recent extension of Bartlett corrections to statistics other than the log-likelihood ratio. In this paper we described some of the main results involving Bartlett and Bartlett-type corrections in a unified framework. Although most of the literature has focused on a particular form of the Bartlett-type correction, we have also considered two other forms which are equivalent to order n^{-1} to that form, and compared them through Monte Carlo simulation. For the cases we considered, one of the alternative forms seemed to be clearly preferable to the other two, and to the unmodified statistic for that matter. We have also presented simulation results that show how the independent variables and the number of nuisance parameters can affect the first order asymptotic approximation to some econometric criteria in regression models.

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