A Score Test for Seasonal Fractional Integration and Cointegration

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Abstract

This paper develops a time domain score statistic for testing fractional integration at zero and seasonal frequencies in quarterly time series models. Further, it introduces the notion of fractional cointegration at different frequencies between two seasonally integrated, I(1) series. In testing problem involving seasonal fractional cointegration, it is argued that the alternative hypothesis is one-sided for which the usual score test is not appropriate. Therefore, based on the ideas in Silvapulle and Silvapulle (1995), a one-sided score statistics is constructed. A simulation study finds that the score statistic generally has desirable size and power properties in finite samples. The score statistics are applied to the quarterly Australian consumption function. The income and consumption series are found to be I(1) at zero and seasonal frequencies and these two series are not fractionally cointegrated at any frequency.

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Introduction

Since the work of Box and Jenkins (1976), the short-range dependent autoregressive integrated moving average (ARIMA) processes $\Phi(B)(1-B)^d x_t = \Theta(B)u_t$ where d is an integer and B is the lag operator, have become popular in the empirical analyses of time series. As a generalization of this type of models to incorporate long-range dependence, Granger and Joyeux (1980) and Hosking (1981) discuss fractionally integrated processes in which the difference parameter d is allowed to be a non-integer. The ARIMA models are stationary if $d < 1/2$ and are invertible if $d > -1$ [see Odaki (1993)]. Therefore, given the value of d, integer differencing may be necessary to achieve stationarity and invertibility conditions. Hosking (1981) further demonstrated that for d in the range $0 < d < 0.5$, the process possesses the long-memory property; long-memory is a characteristic of time series in which the dependence between distant observations is not negligible. Such models were found to be useful in modelling economic and financial time series and in providing superior long-run forecasts to the ARMA models, since they have distinct properties from short-memory models, particularly at the lower frequencies [see Granger and Joyex (1980) and Ray (1993) for details].

Recently, estimation of and testing for the d-parameter at zero frequency have been given considerable attention [see the recent survey, Baillie (1995)]. For testing procedures, see Lo (1991), Agiakloglou and Newbold (1994) and the references there in. For estimation procedures, see, Geweke and Porter-Hudak (1983), Fox and Taqqu (1986), Li and McLeod (1986), Sowell (1992), Beran and Terrin (1994) among others.

The main objective of this study is to develop a time-domain score procedure for testing fractional integration at zero and seasonal frequencies in quarterly time series models. Although the importance of seasonal fractionally integrated processes has been realized for some time now [see Porter-Hudak (1990)], the research on this area is still in its infancy.

Porter-Hudak (1990) has studied the properties of seasonal filter $(1-B^4)^d$ where d is not an integer, and demonstrated its importance in analyzing quarterly time series data. On the other hand, Hassler (1994) argues that this filter is fairly rigid in the sense that the contributions of half yearly and yearly oscillations and of long-run behavior are all governed by one common d. He further argues that the importance of fractional integration at seasonal frequencies can be separated by means of a flexible filter of the form $(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}$ where d_1, d_2 and d_3 may be non-integers. This class of models appears to be useful as it is a natural generalization of the seasonally integrated models, introduced by Hylleberg *et al*. (1990), that are popular in analyzing quarterly time series. Hassler (1994) uses regression of the periodogram to estimate and test for parameters d_1, d_2 and d_3. To our knowledge, this is the only study available in the literature on fractional integration.

In this paper, we develop a time-domain score statistic for testing fractional integration at zero and seasonal frequencies. Robinson (1994) proposed a frequency-domain score statistic for testing fractional integration in a general model which includes I(1), I(2), cyclic and quarterly I(1) and others. While the Robinson's test can be applied [see Silvapulle (1995)] the time-domain score test is computationally attractive, particularly for the current testing problem and it is likely to be preferred by researchers unused to frequency domain approach. The score-type tests are very popular in econometrics for two reasons: Firstly, only the model under the null hypothesis needs to be estimated which is considerably simpler than the model under the alternative hypothesis where as other tests such as likelihood ratio and Wald-type are generally difficult to apply because the full model under the alternative hypothesis needs to be estimated. Secondly, the score test is asymptotically equivalent to the likelihood ratio test.

We introduce the idea of fractional cointegaration at different frequencies between two integrated variables. In testing problem concerning seasonal fractional cointegration, the alternative hypothesis is one-sided; how its one-sided nature arises will be explained in section 4. Further, based on the ideas in Silvapulle and Silvapulle (1995), we construct a one-sided score test, because two-sided test is not appropriate and a test incorporating one-sided nature of the alternative hypothesis is in general more powerful than its two-sided counterpart.

This article is planned as follows: The model is specified and a score test is developed in section 2. A procedure to construct a one-sided score test is outlined in section 3. Seasonal fractional cointegration is introduced in section 4. A simulation experiment conducted to evaluate the test's finite sample properties in a number of cases and its results are discussed in Section 5. In section 6, the score tests are applied to the Australian consumption function. Some concluding remarks are made in section 7.

2. The Model and a Score Test for Unrestricted Alternatives

Consider the stationary seasonal fractionally integrated ARIMA (p,d,q) process,

$$\Phi(B)(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}x_t = \Theta(B)e_t,\ t=1,2,...,T \qquad (1)$$

where $\Phi(B) = 1-\phi_1 B-\phi_2 B^2 - ... - \phi_p B^p$, $\Theta(B) = 1+\theta_1 B + \theta_2 B^2 + ... + \theta_q B^q$,

$\Phi = (\phi_1,...,\phi_p)$ is a set of unknown AR parameters, $\Theta = (\theta_1,...,\theta_q)$ is a set of unknown MA parameters, B is the lag operator defined as $Bx_t = x_{t-1}$, e_t is white noise with $var(e_t) = \sigma^2$ and d_1, d_2 and d_3 are unknown fractionally integrated parameters at 0 and seasonal frequencies 1/2 and 1 respectively. It is assumed that $-1.0 < d_i < 0.5$ for i=1,2,3 to ensure the stationarity of the process. The x_t may be observable time series or the unobservable error term in the regression,

$$y_t = z_t\beta' + x_t \tag{2}$$

where z_t is a 1xk vector of stochastic or non-stochastic variables and β is a 1xk vector of unknown parameters.

Let $\gamma = (d, \eta)$ where $d = (d_1, d_2, d_3)$ and $\eta = (\Phi\ , \Theta, \beta, \sigma^2)$. The null and alternative hypotheses of interest are

H_0: d = **0**

and

H_1: d ≠ **0**

respectively, where **0** = (0, 0, 0). The score statistics for testing H_0 against H_1 takes the form,

$T_0 = n^{-1}s_d' \iota_{dd}^{-1} s_d$ where n = (T-2m), m is the number of terms in the expansion of log(1+B), s_d and L_{dd} are the slope of the log likelihood fuction L(.) of T observations and the information matrix of s_{dd} respectively, derived in the Appendix 1. In section 3, we will assess the sensitivity of the statistic for the various choice of m.

The score statistic developed in this section can be used to test (i) the null of stationary short-memory ARMA model against stationary fractional alternatives and (ii) the null of unit roots at zero and seasonal frequencies against nonstationary seasonal fractional alternatives. The time-domain statistics can easily be developed for testing seasonal fractional integration in models such as $(1-B)^{d_1}(1-B^{12})^{d_2}$ and $(1-B)^{d_1}(1-B^4)^{d_2}$.

3. A Score Test for One-Sided Alternatives

In this section, we will explain how a one-sided score statistic can be constructed. One-sided test is useful when the parameter space under alternative hypothesis can be restricted using prior knowledge or otherwise. An application of the two-sided statistic for such testing problem can result in model misspecification and subsequently, misleading conclusions. Silvapulle and Silvapulle (1995) have developed a procedure whereby one-sided score statistic can be constructed from its two-sided version and have shown that the one-sided statistic has asymptotically weighted sum of chi-sqared distributions, known as chi-bar squared distribution under the null hypothesis. We briefly outline this procedure in order to construct one-sided score statistic from its two-sided version proposed in the previous section.

Suppose that the null and alternative hypotheses of interest are

H_0: d = **0**

and

K: d ≥ **0**

respectively, where d ≥ **0** is interpreted coordinatewise. In the previous section, we derived the score vector and the test statistics for testing H_0: d = **0** against H_1: d ≠ **0** as $s_d = \frac{\partial L(0,\tilde{\eta})}{\partial d}$ and $T_0 = n^{-1} s_d' \iota_{dd}^{-1} s_d$ where n = (T-2m), respectively. Now, using the result that $n^{-1/2} s_d \sim N(\iota_{dd} d, \iota_{dd})$ under H_1 for small d and following Silvapulle and Silvapulle (1995), we define the score test statistic for H_0 against K as

$$T_s = [U'\iota_{dd}U - \inf\{(U-d)'\iota_{dd}(U-d): d \geq \mathbf{0}\}] \tag{5}$$

where $U = n^{-1/2}\iota_{dd}^{-1}s_d$. Silvapulle and Silvapulle (1995) have shown that under H_0, T_s has a chi-bar squared distribution. The p-value for rejecting H_0 can be computed as

$$\Pr\ (T_s \geq c) = \sum_{i=1}^{3} w_i\ \Pr(\chi_i^2 \geq c),$$

where $w_1 = \{(1/4)\pi^{-1}(2\pi - \arccos(\rho_{12}) - \arccos(\rho_{13}) - \arccos(\rho_{23}))\}$,

$w_2 = \{(1/4)\pi^{-1}(3\pi\text{-}\arccos(\rho_{12.3})\text{-}\arccos(\rho_{13.2}) - \arccos(\rho_{23.1}))\}$

and $w_3 = (1/2 - w_2)$,

$(\rho_{12}, \rho_{13}, \rho_{23})$ and $(\rho_{12.3}, \rho_{13.2}, \rho_{23.1})$ are sample correlation and partial correlation coefficients respectively, corresponding to information matrix ι_{dd}. These weights are given in Wolak (1987).

Note that to compute the one-sided T_s statistic what is required is only the two-sided statistic developed in section 2. Once T_0 has been computed, then inf{.} in (5) cab be computed using a quadratic program [see for example, QOPROG and NCONF in IMSL].

The one-sided statistic developed in this section can be used for testing the null of short-memory ARMA process against the long-memory alternatives; ie testing H_0 against K in (1). In what follows it will be shown that in testing (i) the null of no-cointegration against fractional cointegration and (ii) the null of cointegaration against the fractional cointegration, the alternative hypotheses are one-sided for which the use of one-sided score statistic is appropriate.

4. Seasonal Fractional Cointegration

In this section, we introduce a generalized notion of seasonal cointegration known as seasonal fractional cointegration and explain how one-sided testing problem can arise in this context.

From the work of Engle and Granger (1987), testing for cointegration as a long-run equilibrium relationship between I(1) series has been studied quite extensively. Until recently, attention has been mainly focused on testing for integration of and cointegration between time series only at zero frequency.

Hylleberg *et al*. (1990) and Engle *et al*. (1993)], introduced the notion of cointegration at different frequencies between time series. To illustrate this, consider a pair of quarterly time series y_t and z_{1t} which are I(1) at zero as well as at seasonal frequencies 1/2 and 1. Denote I(1) at frequencies 0, 1/2 and 1 by I(**1**) where **1** =(1, 1, 1), and I(0) at these frequencies by I(**0**) where **0** = (0, 0, 0); similarly, denote cointegration between y_t and z_{1t} of order CI(1, 0) at 0, 1/2 and 1 by CI(**1, 0**), and CI(1, 1) at all three frequencies by CI(**1, 1**). Now, in general the linear combination $m_t = \alpha_1 y_t + \alpha_2 z_{1t}$ is also I(**1**). However, if there exist a vector (α_1, α_2) which is common to all frequencies such that m_t is I(**0**), then y_t and z_{1t} are said to be cointegrated of order CI(**1, 1**). In the empirical analyses of quarterly time series, testing for m_t is I(**0**) or I(**1**), equivalently, the order of cointegration is CI(**1, 1**) or CI(**1**, **0**), has become important in order to model the long-run equilibrium relationships.

Granger (1986) argued that two series can be cointegrated with order of cointegration between 1 and 0; this is known as fractional cointegration. In our illustrative model, this means that m_t is I(d) where d = (d_1, d_2, d_3) and $0 < d_i < 1$ for some i. Testing for fractional cointegration only at zero frequency has been studied by Cheung and Lai (1993).

The usual choice that m_t is I(**0**) or I(**1**) represents extreme situations. The main aim of the previous studies has been to establish whether the equilibrium deviation m_t is mean-reverting or mean-averting. As has been argued in Hosking (1981), fractionally integrated processes also possess mean-reverting property. When **0.5** < d < **1** where **0.5** = (0.5, 0.5, 0.5), m_t is covariance non-stationary, because its variance is not finite [see Hosking (1981)]. However, it is slowly mean-reverting, because the innovation has no permanent effect on m_t. By contrast, the effect of an innovation on the I(1) process, which is both covariance non-stationary and mean-averting, can persist forever. Therefore, testing the null hypothesis that m_t is I(**1**) against m_t is I(d) where **0.5** < d < **1**, is important for correct specification of equilibrium relations. Moreover, that the stationary equilibrium deviation m_t can also possess the long-memory property is not captured in the I(0) process. Therefore, testing the null hypothesis that m_t is short-memory stationary against that m_t is I(d) where **0** < d < **0.5**, is equally important.

In order to define the hypotheses of interest formally, let (α_1, α_2) = (1, -δ) and $y_t = \delta z_{1t} + u_{1t}$, where u_{1t} is the error term. Consider the following process:

$$(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}\, u_{1t} = e_{1t} \qquad (7)$$

where d_1, d_2 and d_3 are unknown parameters. Now, assume that the order of cointegration corresponding to 0, 1/2 and 1 frequencies are b_1, b_2 and b_3 respectively, denoted by CI(**1**, b) where b = (b_1, b_2, b_3). Therefore, $d_i = 1- b_i$, i =1, 2, 3. In (7), $d_1 = d_2 = d_3 = 0$ implies that y_t and z_{1t} are cointegrated at zero and seasonal frequencies of order CI(**1, 1**). On the other hand, $d_1 = d_2 = d_3 = 1$ implies that they are not cointegrated at any frequency. In other words, the order of cointegration is CI(**1**, **0)**. These two cases have been studied in the literature.

Clearly, from the foregoing brief over view of the literature, a natural generalization of cointegration is the utility of the process (7) for $\mathbf{0} \leq b \leq \mathbf{1}$. To our knowledge, this has not been studied in the literature. So, let us assume that $\mathbf{0} \leq b \leq \mathbf{1}$. For such values of b, y_t and z_{1t} may be fractionally cointegrated at some frequencies. Therefore, testing for the order of cointegration being CI(**1**, **1**) against fractional cointegration at some frequencies entails testing H_0: $d = \mathbf{0}$ against K: $d \geq \mathbf{0}$. On the other hand, testing for the order of cointegration being C1(**1**, **0**) against fractional cointegration at some frequencies entails testing H_{01}: $d = \mathbf{1}$ against K_1: $d \leq \mathbf{1}$. In both testing problems, the alternative hypotheses are one-sided involving a vector parameter.

5. Simulation Experiment and the Results

A Monte Carlo simulation experiment was conducted to assess the finite sample properties of the score tests under various conditions, using the asymptotic chi-squared critical values with 3 degrees of freedom at 1, 5 and 10 percent significance levels. Sample sizes T=52, 100 and 252 are chosen. The test statistics are computed for m = 5, 10 and 15 where m is the number of terms chosen from the expansion of $\ell og(1+B)$. Two- and one-thousand replications are used for size and power calculations respectively. For all computations GAUSS programming software was used.

5.1 Experiment

Experiment 1: The properties of the T_0 procedure for testing H_0: $d = \mathbf{0}$ against H_1:$d \neq \mathbf{0}$ are assessed. Under H_0 and H_1, all parameters in η except σ^2 are assumed to be zero. The process $x_t = e_t$ where e_t is N(0,1) random variable, is generated under H_0. Fractionally integrated process $(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}x_t$ is generated for d_i=0.1, 0.2, 0.3 and 0.4 for i= 1, 2 and 3, under H_1. When $d_1 = d_2 = d_3$, using the expressions for autocovariances of the seasonal fractionally integrated process x_t given in the Appendix, the desired TxT covariance matrix Σ is constructed. The process x_t is generated as $x_t = Pe_t$ where P is the lower triangular Choleski decomposition of Σ, under H_1. When not all d's are equal, the flexible seasonal filter is generated using the formulae (i) to (vi) given in Appendix. The sizes and powers of the T_0 test are computed at selected combinations of d's and reported in Table 1.

Experiment 2: The size and power properties of the T_0 test are explored in the presence of AR(1) under both H_0 and H_1. The stationary AR(1) process $(1-\phi_1 B)x_t = e_t$ is generated under H_0 for ϕ_1 = 0.1, 0.2, ..., 0.9. Initial observations are adjusted to ensure the stationarity of the AR(1) process. The maximum likelihood procedure is used to estimate the AR(1) parameter under H_0. Under H_1, first, the seasonal fractionally integrated process is generated as in Experiment 1 and then it is transformed to the AR(1) process. The sizes and powers of the T_0 for testing H_0: $d = \mathbf{0}$ against H_1: $d \neq \mathbf{0}$ in the presence of AR(1) are reported in Table 2; only the figures corresponding to ϕ_1 = 0.3 to 0.7 are given.

Experiment 3: This involves testing for fractional cointegration of two variables that are integrated at seasonal frequencies 0, 1/2 and 1. Our design of the Monte Carlo simulation for this experiment is similar to that of Engle and Granger (1987). Models considered are:

$$y_t + z_{1t} = u_{1t} \tag{8}$$

$$y_t + 2z_{1t} = u_{2t} \tag{9}$$

$$u_{1t} = (1-B^4)e_{1t} \tag{10}$$

$$u_{2t} = (1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}\, e_{2t} \tag{11}$$

where e_{1t} and e_{2t} are generated as N(0, 1) variates. The model $y_t = \delta z_{1t} + u_{2t}$ is estimated by OLS method and the null hypothesis that $d = \mathbf{0}$ in (11) is tested against the alternative hypothesis that $d \neq \mathbf{0}$ with u_{2t} replaced by OLS residuals. The powers are computed for some selected combinations of d_i = 0.1, 0.2, 0.3, 0.4 for i = 1,2, 3. The sizes and powers of T_0 are given in Table 3.

5.2 Results

It is clear from Table 1 that T_0 has desirable size and power properties. When T=52, the sizes for m=5 are close to the nominal level and they increase marginally as m increases. When T=100 or 252, these sizes do not appear to depend on m. The powers of this test increase sharply as d_i, i = 1, 2, 3, and/or T increase.

In the presence of AR(1), the T_0 test tends to have smaller sizes than the nominal level, and they appear to be stable as ϕ_1 increases from 0.1 to 0.9 (see Table 2). The powers of this test is almost equal to the sizes when values of d_i is 0.1 for i = 1, 2, 3 and T = 52, and they increase as d_i and/or T increase. Although the powers of the test are smaller in small samples, compared to those in the absence of AR(1), when the sample size is large these power differences are not noticeable.

The results of the T_0 test of cointegration against fractional cointegration in Table 3 reveal that the sizes are close to the nominal level for m = 5 in all samples, and they increase as m increases. The T_0 test has good powers against the fractional cointegration. We also assessed the T_0 test's size and power properties for testing no-cointegration against fractional cointegration. The sizes in this case are close to the nominal level for all m and T values and the test has good powers for T which is lager than 52. The results of this testing are not reported but are available from author on request.

The properties of the tests described above are also observed at 1 and 10 per cent levels for all T.
In summary, the statistics have desirable finite sample size and power properties in all most all cases studied. For testing the null of stationary ARMA against fractionally integrated alternatives, the number of terms, m, in the logarithmic expansion should be carefully chosen, particularly when the sample size is small. When the sample size is large, m can be chosen somewhat arbitrarily; however, for higher values of m the test seems to have better power.

6. An Illustrative Example

We consider the model,

$$y_t = \delta z_{1t} + u_t$$

where y_t and z_{1t} are logarithms of quarterly Australian real total consumption of non-durables and real disposable income in 1990 prices, respectively. Data for y_t and z_{1t} were collected from the DX Data Base at La Trobe university; they span the period 1959Q3 to 1994Q4. The sample size is 142.

We test for unit roots in y_t and z_{1t} at seasonal frequencies against the fractional integration at some frequencies using the two-sided T_0 statistic, and then test whether y_t and z_{1t} are fractionally cointegrated at some frequencies using the one-sided score statistic.

Consider the models,

$$(1-\phi_1 L-\dots-\phi_p L^p)(1-B)^{1+d_1}(1+B)^{1+d_2}(1+B^2)^{1+d_3}\, y_t = e_{1t} \tag{12}$$

and

$$(1-\phi_1 L-\dots-\phi_p L^p)(1-B)^{1+d_1}(1+B)^{1+d_2}(1+B^2)^{1+d_3}\, z_{1t} = e_{2t}. \tag{13}$$

First, we test whether the series is I(**1**) or I(**1**+d). This is equivalent to testing H_0: d = **0** against H_1: d ≠ **0** in (12) and (13). From the estimated sample autocorrelation functions and partial autocorrelation functions of the $(1-B^4)y_t$ and $(1-B^4)z_{1t}$ series, the order of the foregoing two autoregressive processes are chosen as p = 5 and p = 7 respectively.

The T_0 statistics for testing H_0: d = **0** against H_1: d ≠ **0** in (12) with p = 5 are computed as 5.82, 2.95 and 1.80 respectively for m = 5, 10 and 15. The calculated statistics for testing H_0 against H_1 in (13) with p = 7 are 4.52, 4.55 and 3.98 respectively for m = 5, 10, and 15. These statistics are not significant at the 5 per cent chi-squared critical value with three degrees of freedom. Therefore, we conclude that both income and consumption series are I(1) at seasonal frequencies 0, 1/2 and 1.

Second, no-cointegration of y_t and z_{1t} is tested against fractional cointegration. Under the permanent income hypothesis, y_t and z_{1t} may be cointegrated with cointegrating vector (1, -1) which is common to all frequencies. Consider the following model,

$$(1-\phi_1 L-\phi_2 L^2-\phi_3 L^3)(1-B)^{1+d_1}(1+B)^{1+d_2}(1+B^2)^{1+d_3}\, u_t = e_t \qquad (14)$$

where $u_t = y_t - z_{1t}$. From the autocorrelation and partial autocorrelation functions of $(1-B^4)u_t$ p = 3 is chosen. We argued in section 4 that the alternative hypothesis for this testing problem is one-sided; the familiar two-sided score statistic is not appropriate. Therefore, we use the one-sided score statistic constructed in section 3 for testing H_0: d = **0** against K_2: d ≤ **0** in (14). In order to compute the one-sided T_s^c statistic, the two-sided score statistic

(T^c_0), inf{.}, and the corresponding weights given in section 3 are computed for m = 5, 10 and 15 and the results are summarized as follows:

	m		
	5	10	15
T^c_0	7.61	5.53	3.75
Inf{.}	4.53	2.61	1.45
weights	(0.04, 0.23 ,0.46)	(0.04, 0.24, 0.46)	(0.04, 0.23, 0.46)
T^c_s	3.08	2.92	2.30
$Pr(T_s > T^c_s)$	0.09	0.12	0.16

where the figures in parentheses are weights, from left, correspond to $\chi_{(3)}$, $\chi_{(2)}$ and $\chi_{(1)}$ distributions respectively. Since the p-values of the one-sided statistic are computed as 0.07, 0.12 and 0.12 for m= 5, 10 and 15 respectively, we do not reject the null hypothesis that there is no cointegration and conclude that the quarterly income and consumption series not cointegrated either at zero or at seasonal frequencies.

7. Conclusion

This paper develops a time-domain score statistic for testing seasonal fractional integration of quarterly time series and considers the testing for seasonal fractional cointegration of two I(1) series. It is argued that proper formulation of the latter testing problem leads to a multiparameter one-sided alternative hypothesis. A one-sided score statistic is constructed, which is generally more powerful in the restricted parameter space than its two-sided counterpart. A simulation study is conducted to assess the properties of the score procedure for testing the following cases: (i) H_0: I(0) process against H_1: fractional integration at some of zero and seasonal frequencies; (ii) the same hypotheses as in (i), but AR(1) is present under both hypotheses; and (iii) cointegration of order **1** against fractional cointegration at zero and seasonal frequencies between two seasonally integrated variables. The results indicate that the tests generally do have desirable size and power properties.

The application of the score statistics are illustrated using the Australian quarterly consumption function. The two-sided statistic is used to test whether both series are I(1) or fractionally integrated and they are found to be I(1) at all frequencies. An application of the one-sided statistic to test whether they are not cointegrated or fractionally cointegrated at some frequencies finds that they are not cointegrated at any frequency.

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Table 1: Probability of rejecting H_0: d = **0** against H_1: d ≠ **0** in $(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}x_t = e_t$, using 5 per cent chisquared critical value with three degree of freedom.

A: Sizes of the T_0 test

(d_1, d_2, d_3)	T	m= 5	10	15
(0.0 0.0 0.0)	52	0.06	0.08	0.09
	100	0.05	0.06	0.06
	252	0.04	0.06	0.05

B: Powers of the T_0 test

(d_1, d_2, d_3)	T	m= 5	10	15
(0.1 0.1 0.1)	52	0.10	0.14	0.19
	100	0.15	0.17	0.18
	252	0.26	0.27	0.29
(0.2 0.2 0.1)	52	0.16	0.20	0.22
	100	0.21	0.27	0.28
	252	0.34	0.37	0.39
(0.3 0.3 0.3)	52	0.44	0.49	0.48
	100	0.39	0.41	0.42
	252	0.50	0.52	0.54
(0.4 0.3 0.3)	52	0.51	0.53	0.57
	100	0.59	0.63	0.63
	252	0.68	0.72	0.73
(0.4 0.4 0.4)	52	0.86	0.88	0.86
	100	0.95	0.96	0.96
	252	0.98	0.99	1.00

Note: T is the sample size, m is the number of terms chosen from the expression of $\ell og(1-B)$ and d = (d_1, d_2, d_3).

Table 2: Probability of rejecting H_0: d = **0** against H_1: d ≠ **0** in $(1-\phi_1 B)(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3} x_t = e_t$, using 5 per cent chisquared critical value with three degree of freedom.

A: Sizes of the T_0 test

		m =	5			10			15		
(d_1, d_2, d_3)	T	ϕ_1 =	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
(0.0 0.0 0.0)	52		0.04	0.04	0.04	0.05	0.05	0.04	0.08	0.08	0.07
	100		0.04	0.04	0.03	0.04	0.03	0.03	0.05	0.04	0.03
	252		0.04	0.03	0.03	0.03	0.03	0.03	0.04	0.03	0.03
B: Powers of the T_0 test											
(0.1 0.1 0.1)	52		0.04	0.04	0.05	0.07	0.05	0.07	0.10	0.08	0.07
	100		0.05	0.03	0.03	0.07	0.04	0.06	0.08	0.06	0.06
	252		0.11	0.10	0.10	0.12	0.11	0.12	0.13	0.14	0.14
(0.2 0.2 0.1)	52		0.08	0.08	0.11	0.12	0.11	0.14	0.16	0.12	0.15
	100		0.11	0.13	0.12	0.18	0.18	0.16	0.20	0.21	0.19
	252		0.28	0.27	0.28	0.30	0.32	0.32	0.34	0.33	0.34
(0.3 0.3 0.3)	52		0.29	0.26	0.30	0.35	0.34	0.34	0.34	0.34	0.34
	100		0.46	0.45	0.47	0.61	0.57	0.53	0.60	0.54	0.53
	252		0.68	0.66	0.66	0.67	0.68	0.67	0.68	0.68	0.68
(0.4 0.3 0.3)	52		0.41	0.40	0.40	0.42	0.43	0.44	0.43	0.42	0.45
	100		0.56	0.57	0.58	0.58	0.57	0.59	0.61	0.60	0.60
	252		0.70	0.71	0.73	0.72	0.72	0.73	0.77	0.75	0.75
(0.4 0.4 0.4)	52		0.73	0.69	0.74	0.78	0.73	0.76	0.74	0.70	0.70
	100		0.90	0.92	0.92	0.95	0.94	0.94	0.94	0.93	0.92
	252		0.98	0.98	0.98	0.99	0.99	1.00	1.00	1.00	1.00

Note: T is the sample size, m is the number of terms chosen from the expression of $\ell og(1-B)$ and d = (d_1, d_2, d_3).

Table 3: Probability of rejecting H_0: d = **0** against H_1: d ≠ **0** in $(1-B)^{d_1} (1+B)^{d_2} (1+B^2)^{d_3} x_t=e_t$, where $\{x_t\}$ are OLS residuals from $y_t = \delta z_t + e_t$, using the 5 per cent chisquared critical value with three degrees of freedom.

A: Sizes of the T_0 test

$(d_1\ d_2\ d_3)$	T	m=	5	10	15
(0.0 0.0 0.0)	52		0.06	0.09	0.12
	100		0.07	0.08	0.10
	252		0.06	0.08	0.10

B: Powers of the T_0 test

$(d_1\ d_2\ d_3)$	T	m=	5	10	15
(0.1 0.1 0.1)	52		0.10	0.11	0.18
	100		0.14	0.16	0.17
	252		0.38	0.45	0.47
(0.20 0.20 .1)	52		0.20	0.25	0.25
	100		0.32	0.41	0.44
	252		0.52	0.58	0.61
(0.3 0.3 0.3)	52		0.48	0.52	0.58
	100		0.62	0.68	0.70
	252		0.80	0.89	0.92
(0.4 0.3 0.3)	52		0.50	0.58	0.61
	100		0.68	0.72	0.75
	252		0.89	0.92	0.92
(0.4 0.4 0.4)	52		0.82	0.88	0.90
	100		0.88	0.89	0.92
	252		0.99	1.00	1.00

Note: T is the sample size, m is the number of terms chosen from the expression of $\ell og(1-B)$ and $d = (d_1, d_2, d_3)$.

Appendix 1

The log of the conditional likelihood, $\ell_t(\gamma)$, of e_t corresponding to (1) and (2) is given as

$$\ell_t(\gamma) = - \ell og(2\pi) - (1/2)\ell og(\sigma) - (1/2)\sigma^{-2}[\Theta^{-1}(B)\Phi(B)(1-B)^{d_1}(1+B)^{d_2}(1+B^2)^{d_3}(y_t - z_t\beta')]^2 \quad (3)$$

and the log likelihood, $L(\gamma)$, for the T observations is given as

$$L(\gamma) = \sum_{t=1}^{T} \ell_t(\gamma).$$

The familiar score statistic for testing H_0 against H_1 takes the form

$$\frac{\partial L(\gamma)}{\partial \gamma} \left[E_{H_0} \left(\frac{\partial L(\gamma)}{\partial \gamma'} \; \frac{\partial L(\gamma)}{\partial \gamma} \right) \right] \frac{\partial L(\gamma)}{\partial \gamma'} \Big|_{d=0,\eta=\tilde{\eta}} \quad (4)$$

where $\tilde{\eta}$ is the maximum likelihood estimator of η under H_0, $\frac{\partial L(\gamma)}{\partial \gamma} \big|_{d=0,\eta=\tilde{\eta}}$ and

$E_{H_0}\left[\left(\partial L(\gamma)/\partial \gamma'\right)\left(\partial L(\gamma)/\partial \gamma\right)\right]$ is the information matrix for (T-2m) observations. Let $\iota_{\gamma\gamma}$ be the information matrix for a single observation which is partitioned as

$$\begin{bmatrix} \iota_{dd} & \iota_{d\eta} \\ \iota_{\eta d} & \iota_{\eta\eta} \end{bmatrix}.$$

The first order derivatives of $L(\gamma)$ with respect to d_1, d_2 and d_3 are

$$\frac{\partial L(\gamma)}{\partial d_1} \bullet (1/\sigma^2) \sum_{t=1}^{T} e_t \sum_{j=1}^{m} j^{-1} e_{t-j},$$

$$\frac{\partial L(\gamma)}{\partial d_2} \bullet -(1/\sigma^2) \sum_{t=1}^{T} e_t \sum_{j=1}^{m} j^{-1} (-1)^{j-1} e_{t-j},$$

and

respectively, where m is chosen such that $\ell og(1+B)$ can be approximated by the first m terms in its expansion.

$$\frac{\partial L(\gamma)}{\partial d_3} \bullet -(1/\sigma^2) \sum_{t=1}^{T} e_t \sum_{j=1}^{m} j^{-1} (-1)^{j-1} e_{t-2j}$$

It may be verified that the lower triangular part of ι_{dd} is

$$\begin{bmatrix} \sum_{j=1}^{m} j^{-2} & & \\ \sum_{j=1}^{m} (-1)^{j-1} j^{-2} & \sum_{j=1}^{m} j^{-2} & \\ \sum_{j=1}^{[m/2]} (-1)^{j-1} (2j^2)^{-1} & \sum_{j=1}^{[m/2]} (-1)^{j} (2j^2)^{-1} & \sum_{j=1}^{m} j^{-2} \end{bmatrix}$$

where [m/2] is the largest integer of m/2. Now, we define the score vector s_d as

$s_d = \left(\frac{\partial L(\gamma)}{\partial d_1}, \frac{\partial L(\gamma)}{\partial d_2}, \frac{\partial L(\gamma)}{\partial d_3} \right)$. Under H_0, $n^{-½}s_d \sim N(0,V_d)$ where $n = (T-2m)$ and

$V_d = \iota_{dd} - \iota_{d\eta} \iota_{\eta\eta}^{-1} \iota_{\eta d}$. It can be easily varified that the elements of $\iota_{d\eta}$ are all zero. Therefore, $V_d =$

ι_{dd}. Thus, $n^{-½}s_d \sim N(0, \iota_{dd})$, hence $n^{-1}s'_d \iota_{dd}^{-1} s_d \sim \chi^2_{(3)}$ under H_0. Therefore, the score statistics for testing H_0

against H_1 takes the form,

$T_0 = n^{-1}s'_d \iota_{dd}^{-1} s_d$.

For testing H_0': $d_1=0$ against H_1': $d_1 \neq 0$; i.e, testing for fractional integration only at zero frequency in (1) with $d_2 = d_3 = 0$, the score statistic T_0' takes the simple form,

$$T_0' = \left(\frac{\partial L(\gamma)}{\partial d_1} \right)^2 \Bigg| \left[\left(\sum_{j=1}^{m} j^{-2} \right) (T-m) \right].$$

The proof of the asymptotic distribution of the score statistic derived in this section follows from Robinson (1994). He developed the frequency domain score statistics for unit roots and various forms of non-stationarity and establishes the null and local limit distributions of the statistics under mild regularity conditions. He also argued that slight modification to the proof of the limit theorems yield the same results for their time-domain counter parts and noted that the proof for the time domain version is somewhat simpler [see section 3 in Robinson (1994)].

Appendix 2

When $d_1 = d_2 = d_3$, (1) becomes

$$(1-B^4)^d x_t = u_t \qquad u_t = \Phi(B)^{-1}\Theta(B)e_t. \tag{a1}$$

When $\phi_1 = ...\phi_p = \theta_1 = ... = \theta_q = 0$, (2) is known as rigid seasonally fractionally integrated noise. In this case the kth autocovariance function = cov(x_t, x_{t-k}) = E (x_t, x_{t-k}) is given as

$$\gamma_{4k} = \Gamma(1-2d)\ \Gamma(d+k)/\Gamma(d)\ \Gamma(1-d)\ \Gamma(1-d+k)\sigma_e^2 \tag{a2}$$

and $\gamma_{4k+m} = 0$ for m =1,2,3. (4) can be used to generate regid seasonal fractionally integrated processes [See Hassler (1994). It is very difficult to derive autocovariance function for the flexible filter (1) with $\phi_1 = ... = \phi_p = \theta_1 = ... = \theta_q = 0$. However, the following formulae can be used to generate flexible filters when not all d's are equal:

$$\text{(i)} \quad (1-B)^{d_1}\ (1+B)^{d_2}\ (1+B^2)^{d_3} = \sum_{j=0}^{\infty} \delta_j B^j = \sum_{j=0}^{p} \delta_j B^j$$

where the coefficients δ_j are defined by convolution of

$$\text{(ii)} \quad (1-B)^{d_1} = \sum_{j=0}^{\infty} a_j B^j \text{ with } a_0 = 1,$$

$$a_j = \frac{j-1-d_1}{j} a_{j-1},$$

$$\text{(iv) compute the coefficients of } (1-B)^{d_1}\ (1+B)^{d_2} = \sum_{j=0}^{\infty} \beta_j B^j,$$

and then compute δ_j as,

(iii) $(1+B)^{d_2} = \sum_{j=0}^{\infty} b_j \, B^{j}$ with $b_0=1$,

$$b_j = \frac{d_2 - j + 1}{j} b_{j-1},$$

(iv) $(1+B)^{d_3} = \sum_{j=0}^{\infty} c_j \, B^{2j}$ with $c_0=1$,

$$c_j = \frac{d_3 - 1 + 1}{j} c_{j-1}.$$

$$\beta_j = \sum_{K=0}^{j} a_{j-k} \, b_K,$$

(vi) $$\delta_j = \begin{cases} \sum_{k=0}^{m} c_{m-k} \, \beta_{2k}, & j=2m \\ \sum_{k=0}^{m} c_{m-k} \, \beta_{2k+1}, & j=2m+1 \quad m=0,1,2,... \end{cases}$$

These values of δ_j can be substituted in (i) and a reasonable approximation for seasonal fractional filter can be obtained for a larger p.

These results will be used to generate ARSFIMA processes in section 3 which involves Monte Carlo experiments.