IMPROVED TEST STATISTICS FOR MULTIVARIATE REGRESSION*

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Abstract. Edgeworth expansions to the null distributions of three classical test statistics in the multivariate regression model were derived by Rothenberg (1977) and Phillips (1984) with the purpose of obtaining size-corrected critical values for such tests. We combine their results with the results of Cordeiro and Ferrari (1991) to obtain corrections to be applied to the test statistics directly. Simulation results are also given.

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1. Introduction

The multivariate regression model generalizes the normal linear regression model in the sense that it allows for more than one dependent variable. Three commonly used testing criteria are the Wald, Lagrange multiplier and likelihood ratio tests. Under mild regularity conditions, their null asymptotic distribution is chi-squared, and this is the distribution that one usually uses to carry out hypothesis tests. However, it may not be a good approximation to the exact null distributions of the test statistics. It has even been shown that there exists a systematic relationship that holds among the three statistics, and this implies that three asymptotically equivalent tests may deliver conflicting inference when applied to the same data set. It is thus important to obtain more reliable tests, that is, tests with superior finite-sample behavior.

Rothenberg (1977) and Phillips (1984) have obtained Edgeworth expansions to the null distributions of the Wald, Lagrange multiplier and likelihood ratio statistics to order n^{-1}, where n is the sample size minus the number of regressors and 'to order n^{-1}' means that terms of order smaller than n^{-1} are ignored. They used this result to obtain corrections that can be applied to the critical values of the tests. In this paper we pursue a different and alternative approach: we show that the Edgeworth expansions in their papers can be used to derive corrections that can be applied to the test statistics themselves. The two approaches are equivalent to order n^{-1} when the null hypothesis is true. We also give simulation results comparing the performance of the uncorrected and corrected tests.

2. The model and test statistics

The multivariate regression (MR) model describes the relationship between G dependent variables and K non-stochastic independent variables, where the dependent variables have a multivariate normal distribution with a mean vector that can be written as a linear combination of the K exogenous variables. Let Y be a $T \times G$ matrix containing the observations on the dependent variables, and write the MR model as

$$Y = X\Pi + U,$$

where X is a $T \times K$ matrix of nonstochastic regressors, Π is a $K \times G$ matrix of unknown parameters and U is a $T \times G$ matrix of random disturbances. Each row of U is assumed to be distributed as multivariate normal with mean zero and (nonsingular) covariance matrix Σ. It is further assumed that there is no dependence among the rows of U and that $\text{rank}(X) = K$. Alternatively, we can write the MR model in vector notation as

$$y = (I_G \otimes X)\pi + u,$$

where $y = \text{vec}(Y)$, $\pi = \text{vec}(\Pi)$, $u = \text{vec}(U)$, I_G is the identity matrix of order G, and $\otimes$ denotes the Kronecker product for matrices.

The null hypothesis of interest is stated as $\text{H}_0 : R\pi = r$, where R is a given $q \times GK$ matrix of rank q and r is a q-vector of specified constants, and the alternative hypothesis is $\text{H}_1 : R\pi \neq r$. An important special case is what Berndt and Savin (1977) call 'the uniform mixed case' where $R = D \otimes H$ and the null hypothesis is then written as $\text{H}_0 : H\Pi D' = E$, where D is a nonsingular $G \times G$ matrix, H is an $a \times K$ matrix of rank a, and E is a given $a \times G$ matrix.

Let $\hat{\Pi}$ and $\tilde{\Pi}$ denote the unrestricted and restricted maximum likelihood (ML) estimators of Π, respectively, $\hat{\pi} = \mathrm{vec}(\hat{\Pi})$, $\hat{\Sigma} = (Y - X\hat{\Pi})'(Y - X\hat{\Pi})/n$ and $\tilde{\Sigma} = (Y - X\tilde{\Pi})'(Y - X\tilde{\Pi})/n$, where $n = T - K$. [Note that $\hat{\Sigma}$ and $\tilde{\Sigma}$ are not the ML estimators of Σ for the unrestricted and restricted models, respectively.] Then, the Wald (W), Lagrange multiplier (LM) and likelihood ratio (LR) statistics for the test of $\mathrm{H}_0 : R\pi = r$ can be written as

$$W = n\,\mathrm{tr}\{\hat{\Sigma}^{-1}(\tilde{\Sigma} - \hat{\Sigma})\}, \tag{1}$$

$$LM = n\,\mathrm{tr}\{\tilde{\Sigma}^{-1}(\tilde{\Sigma} - \hat{\Sigma})\}, \tag{2}$$

$$LR = n \log|\hat{\Sigma}^{-1}\tilde{\Sigma}|, \tag{3}$$

respectively.

3. Size-corrected critical values

Let $B = [R(\Sigma \otimes (X'X)^{-1})R']^{-1/2}R$, and partition B as $B = [B_1 \cdots B_G]$, where B_i is a $q \times K$ matrix of coefficients relating to the ith column of Π. Also, define $V = \sqrt{n}(\hat{\Sigma} - \Sigma)$. This matrix has mean zero and covariances $\mathrm{cov}(v_{ij}, v_{kl}) = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$. Finally, let $A_{ij} = B_i(X'X)^{-1}B_j'$ and

$$\alpha = \sum_{ijkl} \sigma_{ik}\sigma_{jl}\,[\mathrm{tr}(A_{ij}A_{kl}) + \mathrm{tr}(A_{ij}A_{lk}) - \mathrm{tr}(A_{ij})\,\mathrm{tr}(A_{kl})],$$

$$\beta = \sum_{ijkl} \sigma_{ik}\sigma_{jl}\,[\mathrm{tr}(A_{ij}A_{kl}) + \mathrm{tr}(A_{ij}A_{lk}) + \mathrm{tr}(A_{ij})\,\mathrm{tr}(A_{kl})].$$

Then, Edgeworth expansions to the null distributions of W, LM and LR may be expressed as (Rothenberg, 1977)

$$\Pr[W \le z] = F_q(z) - \frac{1}{2n}[\alpha f_{q+2}(z) + \beta f_{q+4}(z)] + o(n^{-1}),$$

$$\Pr[LM \le z] = F_q(z) - \frac{1}{2n}[\alpha f_{q+2}(z) - \beta f_{q+4}(z)] + o(n^{-1}),$$

$$\Pr[LR \le z] = F_q(z) - \frac{1}{2n}[\alpha f_{q+2}(z)] + o(n^{-1}),$$

respectively, where $F_q(\cdot)$ is the cumulative distribution function of a chi-squared random variable with q degrees of freedom and $f_{q+r}(\cdot)$ is the density function of a chi-squared random variable with $q+r$ degrees of freedom; see also Phillips (1984).

It is thus possible to obtain size-corrected critical values to order n^{-1} (*i.e.*, when terms of order smaller than n^{-1} are ignored) for the test of $\mathrm{H}_0 : R\pi = r$ by inverting the Edgeworth expansions above. Such corrected critical values are given by

$$z_W^* = \bar{z}\left[1 + \frac{\alpha}{2nq} + \frac{\beta}{2nq(q+2)}\bar{z}\right], \tag{4}$$

$$z_{LM}^* = \bar{z}\left[1 + \frac{\alpha}{2nq} - \frac{\beta}{2nq(q+2)}\bar{z}\right], \tag{5}$$

$$z_{LR}^* = \bar{z}\left[1 + \frac{\alpha}{2nq}\right], \tag{6}$$

where $\bar{z}$ is the critical value for a chi-squared distribution with q degrees of freedom, *i.e.*, $\Pr[\chi^2_q \geq \bar{z}] = \zeta$, where ζ is the nominal size of the test. In the uniform mixed case, (4)-(6) reduce to

$$z^*_W = \bar{z}\left[1 + \frac{G-a+1}{2n} + \frac{G+a+1}{2n(aG+2)}\bar{z}\right],$$
$$z^*_{LM} = \bar{z}\left[1 + \frac{G-a+1}{2n} - \frac{G+a+1}{2n(aG+2)}\bar{z}\right],$$
$$z^*_{LR} = \bar{z}\left[1 + \frac{G-a+1}{2n}\right].$$

Improved tests can then be carried out by comparing the values of three test statistics to the size-corrected critical values presented above.

4. Improved test statistics

Let S_1, S_2 and S_3 represent the W, LM and LR statistics, respectively, and write the asymptotic expansions given in the previous section as

$$\Pr[S_j \leq z] = F_q(z) + \frac{1}{n}[\gamma_{j1} f_{q+2}(z) + \gamma_{j2} f_{q+4}(z)] + o(n^{-1}), \tag{7}$$

where $j = 1, 2, 3$. Using this notation, we have that

$$\gamma_{11} = -\frac{\alpha}{2}, \qquad \gamma_{12} = -\frac{\beta}{2};$$
$$\gamma_{21} = -\frac{\alpha}{2}, \qquad \gamma_{22} = \frac{\beta}{2};$$
$$\gamma_{31} = -\frac{\alpha}{2}, \qquad \gamma_{32} = 0.$$

We shall make use of the following relations $zf_q(z) = [F_q(z) - F_{q+2}(z)]q/2$, $z^2 f_q(z) = [F_{q+2}(z) - F_{q+4}(z)]q(q+2)/2$, $qf_{q+2}(z) = zf_q(z)$ and $q(q+2)f_{q+4}(z) = z^2 f_q(z)$ to rewrite (7) as

$$\Pr[S_j \leq z] = F_q(z) + \frac{1}{n}\sum_{l=0}^{2} \psi_{jl} F_{q+2l}(z) + o(n^{-1}),$$

where

$$\psi_{10} = \frac{\gamma_{11}}{2} = -\frac{\alpha}{4}, \qquad \psi_{11} = \frac{\gamma_{12} - \gamma_{11}}{2} = \frac{\alpha - \beta}{4}, \qquad \psi_{12} = -\frac{\gamma_{12}}{2} = \frac{\beta}{4};$$
$$\psi_{20} = \frac{\gamma_{21}}{2} = -\frac{\alpha}{4}, \qquad \psi_{21} = \frac{\gamma_{22} - \gamma_{21}}{2} = \frac{\alpha + \beta}{4}, \qquad \psi_{22} = -\frac{\gamma_{22}}{2} = -\frac{\beta}{4};$$
$$\psi_{30} = \frac{\gamma_{31}}{2} = -\frac{\alpha}{4}, \qquad \psi_{31} = \frac{\gamma_{32} - \gamma_{31}}{2} = \frac{\alpha}{4}, \qquad \psi_{32} = -\frac{\gamma_{32}}{2} = 0.$$

It can be shown that

$$S^*_j = S_j\left[1 - \frac{2}{n}\sum_{i=1}^{2}\left(\sum_{l=i}^{2}\psi_{jl}\right)\mu'^{-1}_i S_j^{i-1}\right],$$

where μ_i' is the ith moment about zero of a chi-squared random variable with q degrees of freedom, is distributed as χ_q^2 when terms of order smaller than n^{-1} are neglected; see Cordeiro and Ferrari (1991, p.581). For the MR model, we then have that

$$S_j^* = S_j \left[1 - 2 \left(\frac{\psi_{j1} + \psi_{j2}}{nq} + \frac{\psi_{j2}}{nq(q+2)} S_j \right) \right].$$

That is, for the three tests considered in this paper, we can write the corrected statistics as

$$W^* = W \left[1 - \left(\frac{\alpha}{2nq} + \frac{\beta}{2nq(q+2)} W \right) \right], \tag{8}$$

$$LM^* = LM \left[1 - \left(\frac{\alpha}{2nq} - \frac{\beta}{2nq(q+2)} LM \right) \right], \tag{9}$$

$$LR^* = LR \left[1 - \frac{\alpha}{2nq} \right]. \tag{10}$$

Note that the correction of the likelihood ratio statistic only requires the multiplication of the original statistic by a scalar. This is know as the Bartlett correction. The 'correction factors' for the Wald and Lagrange multiplier statistics are function of the unmodified statistics, and such corrections as usually referred to as Bartlett-type corrections.

For the special case of uniform mixed restrictions, (8)-(10) reduce to

$$W^* = W \left[1 - \left(\frac{G-a+1}{2n} + \frac{G+a+1}{2n(aG+2)} W \right) \right],$$

$$LM^* = LM \left[1 - \left(\frac{G-a+1}{2n} - \frac{G+a+1}{2n(aG+2)} LM \right) \right],$$

$$LR^* = LR \left[1 - \frac{G-a+1}{2n} \right].$$

The improved test statistics given above are distributed as χ_q^2 to order n^{-1} under the null hypothesis. Therefore, they are expected to have better size properties in small samples than the uncorrected ones. When the corrections involve unknown parameters, they should be replaced by their null estimates. Finally, the test based on the corrected statistic and the test based on the corrected critical value are equivalent to order n^{-1}, *i.e.*, $\Pr[S_j^* \le \bar{z}] = \Pr[S_j \le z_j^*] = \Pr[\chi_q^2 \le \bar{z}]$, $j = 1, 2, 3$, when terms of order small than n^{-1} are neglected and the null hypothesis is true. Finally, it is possible to show using Cox and Reid's (1987) results that under local alternatives $\Pr[S_j^* > \bar{z}] = \Pr[S_j > \bar{z}]$, $j = 1, 2, 3$, when terms of order smaller than $n^{-1/2}$ are ignored.

5. Simulation results

The model used in the simulation experiment presented here is the multivariate regression model introduced in Section 2 with $K = 5$ and $G = 2$. Two cases are considered: (i) exclusion of a variable ($q = 2$), and (ii) exclusion of two variables ($q = 4$). When generating the data under the null hypothesis, all parameters in π were set equal one, except the ones under restriction which were set equal zero. The regressors were chosen as random draws from a U(0, 1) distribution and kept constant throughout the experiment. The errors were drawn from a standard normal distribution and the number of replications in all simulations was 10,000. The rejection rates for the uncorrected and corrected tests for $q = 2$ and $q = 4$ are given in Tables 1 and 2, respectively, for $n = 10, 20, 40, 60, 80, 100$ and nominal sizes (ζ) 10%, 5% and 1%. Here, '+' denotes the correction based on transformed critical values and '$*$' denotes the correction based on transformed test statistics.

Table 1: $q = 2$

n	ζ	W	W^+	W^*	LR	LR^+	LR^*	LM	LM^+	LM^*
10	10	26.30	15.56	0.00	15.73	24.85	10.10	0.66	0.00	5.11
	5	20.09	10.12	0.00	9.20	23.11	5.09	0.00	0.00	0.39
	1	12.46	4.32	0.00	2.73	35.04	1.09	0.00	0.00	0.00
20	10	14.97	10.55	7.95	11.52	13.71	9.93	7.68	6.11	9.40
	5	9.40	5.61	2.17	5.99	8.64	4.89	2.61	1.96	4.42
	1	3.38	1.43	0.00	1.42	3.82	1.09	0.11	0.05	0.69
40	10	12.75	10.38	9.95	11.04	12.10	10.17	9.26	8.67	10.07
	5	6.99	5.16	4.83	5.52	6.54	5.07	4.17	3.73	4.97
	1	1.96	1.12	0.87	1.21	1.89	1.04	0.56	0.49	0.97
60	10	11.61	10.31	10.18	10.66	11.22	10.28	9.65	9.18	10.22
	5	6.10	5.11	4.95	5.28	5.89	5.03	4.42	4.22	4.99
	1	1.69	1.13	1.00	1.18	1.63	1.11	0.68	0.63	1.08
80	10	11.22	10.22	10.14	10.51	10.95	10.22	9.77	9.43	10.17
	5	6.14	5.29	5.21	5.47	5.97	5.26	4.80	4.57	5.23
	1	1.37	0.98	0.94	1.00	1.30	0.96	0.77	0.75	0.95
100	10	10.59	9.90	9.86	10.13	10.36	9.87	9.56	9.38	9.87
	5	5.32	4.71	4.64	4.85	5.20	4.66	4.46	4.22	4.65
	1	1.08	0.78	0.75	0.82	1.03	0.78	0.64	0.61	0.76

It is clear from the figures in Tables 1 and 2 that: (i) The rejection rates for W are greater than those for LR and these are greater than those for LM, which was expected given the inequality $W \geq LR \geq LM$; (ii) Among the uncorrected tests, LR had the best performance, with W over-rejecting and LM under-rejecting the null hypothesis; (iii) the correction we proposed (*i.e.*, the correction applied to the test statistics) was more effective in bringing the estimated sizes closer to their nominal levels than the correction based on transformed critical values. In particular, when applied to the critical values of the LR test the latter even caused a deterioration in the performance of the test. For example, for $q = 4$ and $n = 10$, the estimated sizes for the nominal levels of 10%, 5% and 1%, respectively, were (in percentages) 13.80, 7.51 and 1.95 for LR, 52.91, 61.82 and 100.00 for LR^+ (*i.e.*, the test based on transformed critical values), and 10.09, 4.95 and 1.02 for LR^* (*i.e.*, the test based on the transformed statistic). In this case, two undesirable features of the transformation of critical values are revealed: these critical values can be an increasing function of the nominal size, and they can also take negative values, thus leading one to

Table 2: $q = 4$

n	ζ	W	W^+	W^*	LR	LR^+	LR^*	LM	LM^+	LM^*
10	10	34.65	18.29	0.00	13.80	52.91	10.09	0.03	0.00	2.34
	5	27.96	12.01	0.00	7.51	61.82	4.95	0.00	0.00	0.48
	1	18.29	5.25	0.00	1.95	100.00	1.02	0.00	0.00	0.00
20	10	18.35	11.02	4.60	10.98	18.80	9.74	4.09	3.42	8.31
	5	11.74	5.70	0.00	5.42	13.22	4.75	1.21	3.42	3.57
	1	4.71	1.37	0.00	1.08	7.14	0.89	0.02	0.02	0.45
40	10	13.32	10.19	9.39	10.27	13.14	9.93	7.23	6.80	9.52
	5	7.54	5.05	4.43	5.14	7.44	4.83	2.88	2.74	4.59
	1	2.16	1.06	0.75	1.07	2.36	1.00	0.32	0.28	0.88
60	10	11.93	9.88	9.63	10.03	11.72	9.80	8.55	8.27	9.70
	5	6.76	5.10	4.88	5.28	6.67	5.07	3.97	3.78	5.05
	1	1.80	0.95	0.84	0.94	1.80	0.91	0.59	0.57	0.88
80	10	11.78	10.51	10.39	10.61	11.64	10.42	9.30	9.09	10.31
	5	6.60	5.43	5.30	5.50	6.53	5.45	4.50	4.35	5.37
	1	1.73	1.12	1.01	1.08	1.74	1.05	0.74	0.71	1.01
100	10	11.45	10.31	10.20	10.39	11.33	10.25	9.41	9.14	10.23
	5	6.12	5.34	5.28	5.41	6.09	5.34	4.59	4.48	5.26
	1	1.53	1.18	1.11	1.15	1.56	1.14	0.83	0.82	1.14

always reject the null hypothesis regardless of the true data generating process.

6. Concluding remarks

This paper addressed the issue of finite-sample corrections to three important test criteria in the multivariate regression model. Rothenberg (1977) and Phillips (1984) obtained Edgeworth expansions to the null distributions of the test statistics, and from such expansions they obtained size-corrected critical values. Here, we presented an alternative approach. We have shown that it is possible to use the asymptotic expansions in their papers to also derive corrections to be applied to the test statistics. Improved tests can be carried out by comparing the uncorrected statistics to size-corrected critical values or by comparing the values of the corrected statistics given in this paper to the tabulated (unmodified) chi-squared critical values. Our simulations results suggest that the latter is in general more effective.

References

Berndt, E.R. and N.E. Savin, 1977, Conflict among criteria for testing hypotheses in the multivariate linear regression model, Econometrica 45, 1263-1277.

Cordeiro, G.M. and S.L.P. Ferrari, 1991, A modified score test statistic having chi-squared distribution to order n^{-1}, Biometrika 78, 573-582.

Cox, D.R. and N. Reid, 1987, Approximations to noncentral distributions, Canadian Journal of Statistics, 15, 105-114.

Phillips, P.C.B., 1984, Finite sample econometrics using ERA's, Journal of the Japan Statistical Society 14, 107-124.

Rothenberg, T., 1977, Edgeworth expansions for some test statistics in multivariate regression, Working Paper IP-225, Center for Research in Management Science, University of California at Berkeley.