Testing the Null of Stationarity in the Presence of Structural Breaks for Multiple Time Series

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First Draft: April 1994
This Version: August 1994

Abstract

This paper introduces various consistent tests for the null hypothesis of stationarity with possibly unknown multiple structural break points against the alternative of nonstationarity that can be applied to multiple as well as univariate time series. These tests can be applied to either partial or pure structural breaks. It is shown that tests for stationarity become divergent when structural breaks are ignored. We show that we can allow a variety of structural breaks for which limiting distributions are derived and tabulated. Finite sample properties are studied by simulation. We also consider multivariate testing strategy and univariate tests and find that multivariate tests are often more powerful than univariate tests.

JEL Code: C32

Key words: Null of stationarity, Structural breaks, *LM* test, Sargan-Bhargava-Durbin-Hausman test, multiple time series.

1 Introduction

A great deal of research has been devoted to testing the unit root hypothesis. Most conventional approaches specify the null to be nonstationary against the alternative of stationarity. However, as suggested by Kwiatkowski, Phillips, Schmidt and Shin [17, hereafter KPSS], a unit root test should at least be accompanied by stationarity tests for confirmatory data analysis. According to KPSS [17], many series that have been claimed originally to be $I(1)$ by Nelson and Plosser [18] appear to be stationary or inconclusive under stationarity testing. Many test procedures are available for testing the null of stationarity against the alternative of non-stationarity; Park and Choi [21], Park [19], Bierens [5], Herce [15], Dejong, Nankervis, Savin and Whiteman [12], Saikkonen and Luukkonen [27], KPSS [17], Tanaka [30], Khan and Ogaki [16], Stock [29], Tsay [31] and Choi [7]. In addition, Choi and Yu [8] provide a general framework in which many of the tests for $I(m)$ against $I(m+k)$ are generated, and Choi and Ahn [9] developed tests for the null of stationarity for multiple time series.

Another challenging approach against the hypothesis of $I(1)$ is that of the structural break. Perron [23] raises this possibility, and suggests that the null hypothesis of unit root be tested against the alternative of stationarity around a broken trend. His findings suggest that most of the economic time series appear to be stationary when there is a one time crash, and the null of a unit-root is rejected for many of the series. Recently, however, Perron [23] has been criticized for assuming that the structural break points are known, and recent research by Banerjee, Lumsdaine and Stock [4], Perron and Vogelsang [24], Christiano [11], Zivot and Andrews [32], to name a few, replace the exogenous breaks with endogenous breaks. Christiano [11] used the bootstrap method to search for a possible break point in U.S. GNP series and tested whether structural breaks result in spurious behavior of time series. He find little evidence for structural break and refute Perron [23]. Zivot and Andrews [32] allow for an unknown structural break and test the unit-root hypothesis against stationarity. They find that there is less evidence against the unit-root hypothesis than in Perron [23]. Amsler and Lee [1] extend unit-root test suggested in Schimidt and Phillips [28] to test the null of a unit root against the alternative of stationarity with structural change.

So far, most unit root tests in the presence of a structural break are designed to test the null of nonstationarity against the alternative of stationarity around time polynomials with structural breaks. There is, however, no procedure available for testing the null of stationarity with a structural break against the alternative of nonstationarity in univariate as well as multiple time series.

The purpose of this paper is to introduce tests for the null of stationarity with multiple structural breaks at possibly unknown break points. In this paper, we suggest test statistics allowing a structural break under the null of stationarity which diverge under the alternative of nonstationarity. The tests are designed to handle univariate as well as multivariate time series. We allow for unknown break points by taking the supremum of the test statistics along the line of Zivot and Andrews [32]. Our test statistics are variants of the tests for the null of stationarity and the null of cointegration suggested by Choi and Ahn [9, 10]. All limiting distributions are represented by the product of a multivariate Brownian bridge with a structural break parameter denoted as λ. We also report simulation results that study the finite sample performance of the tests. In addition, we will compare the strategy of applying the univariate tests many times and that of using the multivariate tests in finite samples.

We can derive some important benefits by testing the null of stationarity with a structural break against nonstationarity. First, we can use the tests for confirmatory data analysis thus avoiding possible misinterpretation of conventional unit root test results. When both tests result in the same conclusion, we can infer the statistical properties with greater confidence. If the tests disagree, we may conclude that the data is not informative along the line of KPSS [17]. Second, stationarity tests avoid the point null hypothesis so that rejecting the null hypothesis can be thought of as evidence in favor of nonstationarity. Third, we may be able to distinguish a stationary series with a broken trend from a nonstationary series.

This paper is organized as follows. Section 2 introduces the model and hypotheses. Section 3 examine the effect of structural break on stationarity tests studied in Choi and Ahn [9]. Section 4 derives the limiting distribution of our tests for general time series with known structural break points. Section 5 extends the tests in Section 4 to the case of unknown break points. Section 5 reports simulation results. Section 6 concludes with a summary and further remarks. All proofs are in the Appendix.

A few words of notation: All the limits are taken as $''T \to \infty''$ unless otherwise specified. Weak convergence is denoted as $'' \Rightarrow''$. Additionally, $''\Delta''$ signifies the usual difference operator. The standard n-vector Brownian motion is written as $''W(r)''$ and $''f_{vv}(\cdot)''$ denotes the spectral density matrix for $\{v_t\}$. The indicator function is represented by $''\iota_i''$. Lastly, $''A^{(i,j)}''$ denotes the $(i, j) - th$ element of the matrix A.

2 The models, hypotheses and assumptions

We consider the system of equations

$$y_t = Ac_t + x_t, \tag{1}$$

where y_t represents an $n \times 1$ vector time series, c_t represents a $(p + 1) \times 1$ vector of time polynomials and A represents an $n \times (p + 1)$ parameter matrix, respectively. Specifically, $c_t = [1, t, \cdots, t^p]'$, with a suitable weight matrix δ_T, $\delta_T^{-1} c_{[Tr]} \to c(r)$ in $D[0, 1]$. Obviously, $\int_0^1 c(r) c(r)' dr$ is non singular and positive definite (see Park [19, 20]). In this case, $\delta_T = diag[1, T, \cdots, T^p]$ and $c(r) = [1, r, \cdots, r^p]'$. Also, we can transform equation (1) as in Choi and Ahn [9, 10] and Choi and Yu [8]. It is because using $\left\{\bar{S}_t = \sum_{j=1}^t \bar{x}_j\right\}$ results in a degenerate asymptotic distribution for the LM tests due to the fact $\sum_{t=2}^T \Delta\bar{S}_t \bar{S}_{t-1}' = \frac{1}{2}\left(\bar{S}_T \bar{S}_T' - \sum_{t=1}^T \Delta\bar{S}_t \Delta\bar{S}_t'\right)$ and $\bar{S}_T = 0$. After summing up equation (1), we have the following:

$$P_t = A g_t + S_t, \tag{2}$$

where $P_t = \sum_{j=1}^t y_j$, $g_t = \sum_{j=1}^t c_j$, and $S_t = \sum_{j=1}^t x_j$.

Our main interest is in testing whether the time series x_t is stationary when there exist multiple structural breaks. That is, we are interested in testing the null hypothesis

$$H_0 : x_t = I(0) \text{ with structural break} \tag{3}$$

against the alternative

$$H_1 : x_t^{(i)} = I(k_i),\ k_i \geq 1 \text{ for some } i. \tag{4}$$

The null hypothesis (3) is equivalent to that every series in the system of equations given by equation (1) is stationary possibly around time trends of proper order with structural breaks. Under the alternative, we allow each element of x_t to have a different order of integration but require that at least one element be nonstationary.

Letting $w_t = x_t$, we assume under the null that w_t satisfies the following assumptions: $x_t = w_t$ under the null; and $\Delta^{k_i} x_t^{(i)} = w_t^{(i)}$ under the alternative, where $\{w_t\}$ is a vector linear process. More specifically, we make the following assumptions regarding $\{w_t\}$:

A1: $w_t = \sum_{i=0}^{\infty} C_i e_{t-i}$.

A2: $\sum_{i=1}^{\infty} i \|C_i\|$.

A3: $\sum_{i=0}^{\infty} C_i \neq 0$.

A4: $\{e_t, \mathcal{F}_t\}$ is a vector martingale difference sequence and $\mathcal{F}_t$ is σ-algebra generated by $\{e_i, i = 1, \cdots, t\}$.

A5: $E\left(e_t e_t' | \mathcal{F}_{t-1}\right) = \Psi$ where Ψ is positive definite

A6: $\sup_{i,t} E\left(\left|e^{(i)}\right|^{2+\delta} | \mathcal{F}_{t-1}\right) < \infty$ for some $\delta > 0$.

A7: $\Omega = 2\pi f_{ww}(0) = \left(\sum_{i=0}^{\infty} C_i\right) \Psi \left(\sum_{i=0}^{\infty} C_i\right)'$ is positive definite, where C_i's are real matrices and $\|C_i\| = \{tr\,(C_i' C_i)\}^{1/2}$.

A stationary and invertible vector ARMA process is a special case of $\{w_t\}$. Under A1, A2, A4, A5 and A6, we have, as in Phillips and Solo [26, p. 985],

$$T^{-1/2} \sum_{t=1}^{[Tr]} w_t \Rightarrow B(r), \tag{5}$$

where $B(r)$ is a Brownian motion with covariance matrix Ω and $[x]$ denotes the integer part of x. Also, extending Hannan and Heyde's [14] results, we have under A1, A4, A5 and an assumption implied by A2 that

$$T^{-1} \sum_{t=1}^{T} w_t w_t' \xrightarrow{p} \Sigma,$$

where $\Sigma = E(w_t w_t') = \sum_{i=0}^{\infty} C_i \Psi C_i'$. A3 is required to ensure that the limiting distribution of the partial sum process in (5) is non-degenerate and to ensure that $\{w_t\}$ does not have an MA unit root. A7 implies that $\sum_{i=1}^{t} w_i$ is not cointegrated under the null hypothesis.

3 The effect of structural breaks on testing the null of stationarity

By defining an appropriate set of parameters A and regressors c_t of equation (1), the null hypothesis under structural breaks could be formulated. The test statistics we are going to consider are those studied in Choi and Ahn [9, 10], which are given below:

$$LM_I = tr\{(T^{-1}\sum_{t=2}^{T} \Delta\tilde{S}_t\tilde{S}_{t-1}' - \tilde{\Omega}_1')\tilde{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^{T} \tilde{S}_{t-1}\Delta\tilde{S}_t' - \tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\},$$

$$LM_{II} = tr\{(\sum_{t=2}^{T} \Delta\tilde{S}_t\tilde{S}_{t-1}' - T\tilde{\Omega}_1')(\sum_{t=2}^{T} \tilde{S}_{t-1}\tilde{S}_{t-1}')^{-1}(\sum_{t=2}^{T} \tilde{S}_{t-1}\Delta\tilde{S}_t' - T\tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\},$$

$$SBDH_I = tr\{(T^{-2}\sum_{t=1}^{T} \tilde{S}_t\tilde{S}_t')\tilde{\Omega}_l^{-1}\},$$

and

$$SBDH_{II} = tr\{(T^{-2}\sum_{t=1}^{T} \bar{S}_t\bar{S}_t')\bar{\Omega}_l^{-1}\},$$

where the bar ($-$) denotes residuals obtained from equation (1) and the tilde ($\sim$) from equation (2). Note that $\tilde{\Omega}_l$, $\tilde{\Omega}_1$ and $\bar{\Omega}$ are consistent estimates of Ω_l and $\Omega_1 = \sum_{t=2}^{\infty} E(w_1 w_t')$, respectively. As in Hannan [13], $\tilde{\Omega}_l$ and $\bar{\Omega}_l$ are defined as

$$\begin{aligned}
\tilde{\Omega}_l &= \sum_{n=-l}^{l} \tilde{C}(n)k(n/l), \\
\bar{\Omega}_l &= \sum_{n=-l}^{l} \bar{C}(n)k(n/l), \\
\tilde{C}(n) &= T^{-1}\sum_{t=2}^{T-n} \Delta\tilde{S}_t\Delta\tilde{S}_{t+n}'
\end{aligned}$$

$$\bar{C}(n) = T^{-1}\sum_{t=2}^{T-n}\Delta\bar{S}_t\Delta\bar{S}'_{t+n}$$

and $k(n/l)$ is a lag window. Analogously, we define $\tilde{\Omega}_1 = \sum_{n=1}^{l}\tilde{C}(n)k(n/l)$.

Regarding the lag truncation number l and the spectral window, we assume

$$A8 : l \to \infty \text{ as } T \to \infty \text{ and } l = O(T^{\delta}),\ 0 < \delta < \frac{1}{2}.$$

This ensures that the spectral density estimates are consistent. Further, we assume for the lag window $k(z)$ that

$$\begin{aligned} A9 \quad &: \quad k(z) \text{ is a continuous, even function with} \\ & \quad k(0) = 1,\ |k(z)| < 1 \text{ and } \int_{-\infty}^{\infty} k^2(z)dz < \infty. \end{aligned}$$

Assumptions A8 and A9 imply that

$$\sum_{n=-l}^{l} k(n/l) = O(T^{\delta}) \text{ as } l, T \to \infty \text{ and } 0 < \delta < \frac{1}{2}. \tag{6}$$

This result will be used to derive the rate of divergence of the test statistics under the alternative and under the null with misspecified time trends.

To understand the impact of a structural break on the stationarity tests, consider the null of stationarity against nonstationarity studied in Choi and Ahn [9]. Suppose there is a one time pure structural break at $T_B = T\lambda$ for $\lambda \in (0, 1)$ following Andrews [3]. Then, equation (1) should be modified as follows.

$$\begin{aligned} y_t &= A_1 c_t + x_t,\ t = 1, \cdots, T\lambda. \\ y_t &= A_2 c_t + x_t,\ t = T\lambda + 1, \cdots, T, \end{aligned} \tag{7}$$

where $\lambda \in (0,1)$ denotes the break point, and $A_1 \neq A_2$. Let $\iota_1 = 1$ if $t \leq T\lambda$ or 0 otherwise and $\iota_2 = 1$ if $t > T\lambda$ or 0 otherwise. The vector of indicator functions is denoted by $\iota = [\iota_1,\ \iota_2]'$. Equation (7) can then be written using the indicator functions as below.

$$\begin{aligned} y_t &= A_1 c_t \iota_1 + A_2 c_t \iota_2 + x_t, \\ &= A d_t + x_t, \end{aligned} \tag{8}$$

where $d_t = [\iota_1 c_t',\ \iota_2 c_t']' = \iota \otimes c_t$ and $A = [A_1\ A_2]$, which is of dimension $n \times 2(p+1)$. This specification is very simple but useful for our purposes. Further, it is easy to formulate various structural breaks by defining the parameter matrix A and the regressors d_t given the structural break point. Test statistics suggested in this paper are obtained from equation (8) though there are possible alternative expressions. Clearly, (1) is misspecified under the null hypothesis of stationarity with structural breaks, and we expect that the omitted deterministic component would be important enough to cause the estimated residuals to be nonstationary, which makes the test statistics diverge. The following theorem states the effect of a one time structural break on tests for stationarity.

Theorem 1. *Suppose that assumptions A1 - A9 hold and that the time polynomial of order p in the regression equation is correctly specified. Then under the null hypothesis with a one time structural break,*

$$\begin{aligned} (i)\ LM_I &= O_p(T^{2(1-\delta)}), \\ (ii)\ LM_{II} &= O_p(T^{1-\delta}), \\ (iii) SBDH_I &= O_p(T^{1-\delta}), \\ (iv) SBDH_{II} &= O_p(T^{1-\delta}), \end{aligned}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) These results indicate that if there exists a structural break we always reject the null of stationarity asymptotically even when x_t is $I(0)$. Therefore, rejecting the null of stationarity does not automatically imply acceptance of the alternative of nonstationarity; it could be an indication of a structural break instead.

(b) The results are consistent with Perron [23] in the sense that a one time structural break could make stationary time series behave as if it were a nonstationary series.

(c) The rates of divergence of the test statistics are the same as those under the alternative of nonstationarity.

(d) The effects are exactly the opposite when conventional unit root tests such as ADF and Z_α are considered. That is, these unit root tests become inconsistent when a series contains a broken trend.

4 Model with Known Structural Break Points

In this section, we will demonstrate how we can effectively allow for various structural breaks and test the stationarity hypothesis.

4.1 Model with a one time structural break

Suppose there is one time structural break at the point $T_B = T\lambda$. The model is given by equation (8). In what follows, we will consider three types of structural breaks: a pure structural break, a partial structural break and a continuous broken trend. All of these types of structural breaks could be allowed in equation (8).

4.1.1 Case 1: pure structural break

A pure structural break is defined as the case in which all the coefficients of the equation change their value at T_B. Let $d_t = (\iota \otimes c_t)$ with dimension $k = 2 \times (p+1)$. Obviously, $(I_2 \otimes \delta_T^{-1}) d_{[Tr]} \to f(r) = (\iota \otimes c(r))$. The limiting distribution of the OLS estimator is given by

$$T^{1/2}(\bar{A} - A)(I_2 \otimes \delta_T) \Rightarrow \int_0^1 dBf'dr(\int_0^1 ff'dr)^{-1} = N(0, \Omega \otimes (\int_0^1 ff'dr)^{-1}). \tag{9}$$

Note that $\int_0^1 ff' = \int_0^1 (\iota\iota' \otimes cc') = diag[\int_0^1 cc'\iota_1, \int_0^1 cc'\iota_2] = diag[\int_0^\lambda cc', \int_\lambda^1 cc']$, which is non singular. For example, when $c_t = [1, t]$, $\int_0^\lambda cc' = \begin{bmatrix} \lambda & \frac{1}{2}\lambda^2 \\ \frac{1}{2}\lambda^2 & \frac{1}{3}\lambda^3 \end{bmatrix}$, and $\int_\lambda^1 cc' = \begin{bmatrix} 1-\lambda & \frac{1}{2}\left(1-\lambda^2\right) \\ \frac{1}{2}\left(1-\lambda^2\right) & \frac{1}{3}\left(1-\lambda^3\right) \end{bmatrix}$.

4.1.2 Case 2: partial structural break

A partial structural break is defined as the case in which some of the coefficients of the equation change their values at T_B. Rearrange $c_t = [c_{1t}, c_{2t}]'$ such that the coefficients of $c_{2t}(m \times 1)$ change their values but those of c_{1t} do not change their values. Let $d_t = [c'_{1t}, c'_{2t}\iota_1, c'_{2t}\iota_2]'$ and $A = [A_1\ A_2\ A_3]$ with dimension $k \times 1$ and $n \times k$, where $k = p + 1 + m$. Letting δ_{1T} and δ_{2T} be appropriate weight matrices such that $\delta_{1T}^{-1} c_{1[Tr]} \to c_1(r)$, $\delta_{2T}^{-1} c_{2[Tr]} \to c_2(r)$ so that $diag[\delta_{1T}^{-1}, \delta_{2T}^{-1}, \delta_{2T}^{-1}] d_{[Tr]} \to f(r) = (c'_1, c'_2\iota_1, c'_2\iota_2)'$. The limiting distribution of the OLS estimator is given by equation (9) with $\int_0^1 ff'dr = \int_0^1 \begin{bmatrix} c_1c'_1 & c_1c'_2\iota_1 & c_1c'_2\iota_2 \\ c_2c'_1\iota_1 & c_2c'_2\iota_1 & 0 \\ c_2c'_1\iota_2 & 0 & c_2c'_2\iota_2 \end{bmatrix} dr$. For example, when $c_t = [1, t]$ with level shift, $c_{1t} = t$ and $c_{2t} = 1$. Then, $\int c_1c'_1 = \frac{1}{3}$, $\int c_1c'_2\iota_1 = \frac{1}{2}\lambda^2$, $\int c_2c'_2\iota_1 = \lambda$, $\int c_1c'_2\iota_2 = \frac{1}{2}\left(1-\lambda^2\right)$ and $\int c_2c'_2\iota_2 = 1 - \lambda$.

4.1.3 Case 3: structural break with continuous restriction

In Case 1 and 2, a jump of trend is allowed. There is, however, a possibility of no sudden jump (or discontinuity); for example, when a set of parameters change their values at T_B, the dependent variables may lie on the same time path with a change of direction. To formulate continuity with a structural break, there must be at least two coefficients that change their values. With a one time structural break, the continuity restriction reduces the number of parameters to be estimated by one compared to Cases 1 and 2. Without loss of generality, assume a partial structural break at λ for c_{2t} as in Case 2. Then the restriction becomes

$$A_1 c_{1T_B} + A_2 c_{2T_B} = A_1 c_{1T_B} + A_3 c_{2T_B}$$

which implies that

$$A_2 c_{2T_B} - A_3 c_{2T_B} = 0.$$

Since $A_2 \neq A_3$, we can solve the restriction for one arbitrary coefficient. Solving the restriction for the first column of A_2, we obtain

$$A_2^{(1)} c_{2T_B}^{(1)} = -\underline{A}_2 \underline{c}_{2T_B} + A_3 c_{2T_B}, \tag{10}$$

where $\underline{A}_2$ is the $n \times (m-1)$ matrix created by deleting $A_2^{(1)}$, the first column of A_2, from A_2 and $\underline{c}_{2t}$ is $(m-1) \times 1$ vector of constants created by deleting $c_{2t}^{(1)}$, the first element of c_{2t}, from c_{2t}. Using the restriction given by equation (10), we express equation (1) as

$$\begin{aligned}
y_t &= A_1 c_{1t} + A_2 c_{2t} + x_t \\
&= A_1 c_{1t} + \underline{A}_2 \underline{c}_{2t} + A_3 c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} - \underline{A}_2 \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} + x_t \\
&= A_1 c_{1t} + \underline{A}_2 (\underline{c}_{2t} - \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)}) + A_3 c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} + x_t \text{ for } t \leq T\lambda, \\
y_t &= A_1 c_{1t} + A_3 c_{2t} + x_t \text{ for } t > T\lambda.
\end{aligned}$$

The regression equation becomes

$$y_t = A_1 c_{1t} + \underline{A}_2 (\underline{c}_{2t} - \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)}) \iota_1 + A_3 (c_{2t} \iota_2 + c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} \iota_1) + x_t$$

$$= \underline{A}d_t + x_t,$$

where $d_t = [c'_{1t},\ \iota_1(\underline{c}_{2t} - \underline{c}_{2T_B}c^{(1)}_{2t}/c^{(1)}_{2T_B})',\ \iota_2 c'_{2t} + \iota_1 c'_{2T_B}c^{(1)}_{2t}/c^{(1)}_{2T_B}]'$. Note that $\underline{A}$ is $n \times k$ with $k = p + m$ which is reduced in dimension by 1 compared to Case 2. The limiting distribution of the OLS estimator is given by equation (9) with replacement of f.

In what follows, we will derive the limiting distributions of the test statistics. As discussed earlier, we cannot use the estimated residuals for LM_I and LM_{II}. In such a case, we formulate the following regression equation by summing up equation (8) over t as in Choi and Ahn [9, 10] and Choi and Yu [8].

$$S^y_t = Ah_t + S_t, \tag{11}$$

where $h_t = \sum_{i=1}^t d_t$ and $S^y_t = \sum_{i=1}^t y_i$. Denoting the residuals $\bar{x}_t = \Delta\bar{S}_t$ and $\tilde{S}_t$ from equations (8) and (11), respectively, the following test statistics will be considered in this paper.

$$LM_I = tr\{T^{-1}\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}'_{t-1} - \tilde{\Omega}'_1)\tilde{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^{T}\tilde{S}_{t-1}\Delta\tilde{S}'_t - \tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}$$

$$LM_{II} = tr\{(\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}'_{t-1} - T\tilde{\Omega}'_1)(\sum_{t=2}^{T}\tilde{S}_{t-1}\tilde{S}'_{t-1})^{-1}(\sum_{t=2}^{T}\tilde{S}_{t-1}\Delta\tilde{S}'_t - T\tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}$$

$$SBDH_I = tr\{(T^{-2}\sum_{t=1}^{T}\tilde{S}_t\tilde{S}'_t)\tilde{\Omega}_l^{-1}\},$$

$$SBDH_{II} = tr\{(T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}'_t)\bar{\Omega}_l^{-1}\},$$

where

$$\bar{\Omega}_l = \sum_{h=-l}^{l}\bar{C}(h)k(h/l),$$

$$\bar{C}(h) = \frac{1}{T}\sum_{t=2}^{T-h}\Delta\bar{S}_t\bar{S}'_{t+h},$$

$$\tilde{\Omega}_l = \sum_{h=-l}^{l} \tilde{C}(h)k(h/l),$$
$$\tilde{C}(h) = \frac{1}{T}\sum_{t=2}^{T-h} \Delta\tilde{S}_t\tilde{S}'_{t+h}.$$

The limiting distributions for the test statistics are presented in the following theorem.

Theorem 2. *Suppose assumptions A1 - A9 hold.*

(a) Under the null hypothesis with a one time structural break at known point $T_B = \lambda T, \lambda \in (0,1)$,

$$(i)\ LM_I \Rightarrow tr\{\int_0^1 d\tilde{W}(r)\tilde{W}(r)' \int_0^1 \tilde{W}(r)d\tilde{W}(r)'\},$$

$$(ii)\ LM_{II} \Rightarrow tr[\int_0^1 d\tilde{W}(r)\tilde{W}(r)'\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'\}^{-1}\int_0^1 \tilde{W}(r)d\tilde{W}(r)'],$$

$$(iii)\ SBDH_I \Rightarrow tr\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'\},$$

$$(iv)\ SBDH_{II} \Rightarrow tr\{\int_0^1 \bar{W}(r)\bar{W}(r)'\},$$

where

$$\tilde{W}(r) = W(r) - \tilde{\psi}h(r),$$

$$\bar{W}(r) = W(r) - \bar{\psi}h(r),$$

and $\tilde{\psi}$ *and* $\bar{\psi}$ *minimize in* L^2 *norm,*

$$\int_0^1 ||W(r) - \tilde{\psi}h(r)||^2,$$

$$\int_0^1 ||dW(r) - \bar{\psi}f(r)||^2,$$

and $h(r) = \int_0^r f(s)ds$.

(b) Under the alternative hypothesis,

$$(i) LM_I = O_p(T^{2(1-\delta)}),$$

$$
\begin{aligned}
(ii) LM_{II} &= O_p(T^{1-\delta}), \\
(iii) SBDH_I &= O_p(T^{1-\delta}), \\
(iv) SBDH_{II} &= O_p(T^{1-\delta}),
\end{aligned}
$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) These tests are consistent against the alternative hypothesis with or without a structural break and have asymptotic power against nonstationarity only. This is because consistently estimated residuals are used to construct the test statistics when there is no structural break.

(b) From Theorem 1, it is obvious that the test statistics studied in the above theorem diverge when there is more than one structural break. However, it is straightforward to extend our formulation to multiple structural breaks using additional indicator functions.

(c) In connection with (b), allowing for more structural breaks is asymptotically safe but it may affect the power performance because of efficiency losses due to the increased number of parameters to be estimated.

(d) The asymptotic distributions and the finite sample performances of the statistics for the case of no structural break ($d_t = c_t$), are reported in Choi and Ahn [9].

4.2 Model with multiple structural breaks

Consider the case with multiple structural breaks at $\lambda = (\lambda_1, \cdots, \lambda_q)$, $\lambda_i \in (0, 1)$, $i = 1, \cdots, q$, for an n-vector time series y_t. Extending model (8) to the general case, we consider the following regression equation. For expositional purposes, assume partial structural breaks. It was shown in Theorem 5 that we can allow pure structural breaks and/or impose continuity restriction. The equation is thus given by

$$y_t = A_1 c_{1t} + B_1 c_{2t} \iota_1 + \cdots + B_q c_{2t} \iota_q + B_{q+1} c_{2t} \iota_{q+1} + u_t, \tag{12}$$

where $\iota_j = 1$ if $T\lambda_{j-1} < t \leq T\lambda_j$, 0 otherwise. $\iota_1 = 1$ if $t \leq T\lambda_1$ and 0 otherwise. $\iota_{q+1} = 1$ if $T\lambda_q < t \leq T$. Letting $d_t = [c'_{1t}, \iota_1 c'_{2t}, \cdots, \iota_q c'_{2t}, \iota_{q+1} c'_{2t}]'$, we have the same equation as (8). Equation (12) also allows q breaks. Hence, the limiting distributions for the OLS estimators is given by the equation (9) with the proper replacement of $f(r)$ and $h(r)$, where the deterministic components satisfy $\delta_T^{-1} d_{[Tr]} \rightarrow f(r) = [c'_1, \iota_1 c'_2, \cdots, \iota_q c'_2, \iota_{q+1} c'_2]'$ and $\int_0^1 f(r) f(r)' \, dr > 0$. Again, we obtain the following theorem.

Theorem 3. *Suppose that assumptions A1 - A9 hold. Then the results of Theorem 2 hold under multiple structural breaks with proper replacement of $f(r)$ and $h(r) = \int_0^r f(s) \, ds$.*

4.3 Empirical Models

In this subsection, we will consider several models with structural breaks that are consistent with Perron [23]. Suppose that there is a one time structural break which could be either a partial structural break or a pure structural break as in Andrews [3]. Without loss of generality, we assume a partial structural break. We will consider models restricted to be continuous at the time of the structural break and models without such a restriction. The models are:

$M(1)$:

$$y_t = a_0 t^0 + a_1 t^1 + \cdots + a_{k-1} t^{k-1} + a_k^1 t^k + \cdots + a_\ell^1 t^\ell + a_{\ell+1} t^{\ell+1} + \cdots + a_p t^p + x_t \text{ for } t \leq T_B,$$

$$y_t = a_0 t^0 + a_1 t^1 + \cdots + a_{k-1} t^{k-1} + a_k^2 t^k + \cdots + a_\ell^2 t^\ell + a_{\ell+1} t^{\ell+1} + \cdots + a_p t^p + x_t \text{ for } t > T_B.$$

$M(2)$: $M(1)$ + continuous at T_B.

Note that $\ell - k + 1$ parameters for $c_{2t} = [t^k, \cdots, t^\ell]'$ change their values at $T\lambda$. Define indicator functions ι_1 and ι_2 such that $\iota_1(t) = 1$ for $t \leq T_B$ (or $r \leq \lambda$) and $\iota_2(t) = 1$ for $t > T_B$ (or $r > \lambda$), respectively. Then the time polynomial is given by

$$d_t = [1, \cdots, t^{k-1}, t^k \iota_1, t^k \iota_2, \cdots, t^\ell \iota_1, t^\ell \iota_2, t^{\ell+1}, \cdots, t^p]'$$

and

$$\begin{aligned} d_t &= [1, \cdots, t^k, t^k(t\iota_1 + T_B\iota_2), t^k(t - T_B)\iota_2, \cdots, t^k(t^{\ell-k}\iota_1 + T_B^{\ell-k}\iota_2), \\ &\quad t^k(t^{\ell-k} - T_B^{\ell-k})\iota_2, t^{\ell+1}, \cdots, t^p]' \end{aligned}$$

for $M(1)$ and $M(2)$, respectively. Hence, under the null, it is possible to interpret equation (8) as a stationary time series with a structural break. Clearly, the $\ell - k + p + 2$ or $\ell - k + p + 1$ dimensional vector sequences of deterministic trend variables d_t satisfies $\delta_T^{-1} d_{[Tr]} \to f(r)$ in $D[0,1]$ with a suitable weight matrix. In particular, f is given by $f(r) = [1, r, \cdots, r^k\iota_1, r^k\iota_2, \cdots, r^\ell\iota_1, r^\ell\iota_2, t^{\ell+1}, \cdots, r^p]'$ and $[1, r, \cdots, r^k, r^k(r\iota_1 + \lambda\iota_2), r^k(r-\lambda)\iota_2, \cdots, r^k(r^{\ell-k}\iota_1 + \lambda^{\ell-k}\iota_2), r^k(r^{\ell-k} - \lambda^{\ell-k})\iota_2, t^{\ell+1}, \cdots, r^p]'$ for $M(1)$ and $M(2)$, respectively. The weight matrices are $diag[1, T, \cdots, T^k, T^k, \cdots, T^\ell, T^\ell, T^{\ell+1}, \cdots, T^p]$ and $diag[1, T, \cdots, T^k, T^{k+1}, T^{k+1}, \cdots, T^\ell, T^\ell, T^{\ell+1}, \cdots, T^p]$ for $M(1)$ and $M(2)$, respectively.

The models studied by Perron [23] are special cases of $M(1)$ and $M(2)$ with $p = 0$ or 1. In particular, we have the following models:

Model 1: pure level shift ($p = 0$)

$$d_t = [\iota_1(t), \iota_2(t)]',$$

$$f(r) = [\iota_1, \iota_2]'$$

Model 2: partial level shift ($p = 1$)

$$d_t = [\iota_1(t), \iota_2(t), t]',$$

$$f(r) = [\iota_1, \iota_2, r]'$$

Model 3: pure level/trend shift under a continuity restriction ($p = 1$)

$$d_t = [1, t - (t - T_B)\iota_2(t), (t - T_B)\iota_2(t)]',$$

$$f(r) = [1, r - (r - \lambda)\iota_2, (r - \lambda)\iota_2]'$$

Model 4: pure level/trend shift without restriction ($p = 1$)

$$d_t = [\iota_1(t), \iota_2(t), t\iota_1(t), t\iota_2(t)]',$$

$$f(r) = [\iota_1, \iota_2, r\iota_1, r\iota_2]'$$

The limiting distributions of the test statistics for the null of stationarity with a one time structural break are reported in the following lemma.

Theorem 4. *Suppose that assumptions A1 - A9 hold. The results in Theorem 2 hold with*

$$\begin{aligned} h &= [r, \frac{r^2}{2}, \cdots, \frac{r^k}{k}, \frac{1}{k+1}(r^{k+1}\iota_1 + \lambda^{k+1}\iota_2), \frac{1}{k+1}(r^{k+1} - \lambda^{k+1})\iota_2, \cdots, \\ &\quad \frac{1}{\ell+1}(r^{\ell+1}\iota_1 + \lambda^{\ell+1}\iota_2), \frac{1}{\ell+1}(r^{\ell+1} - \lambda^{\ell+1})\iota_2, \cdots, \frac{r^{p+1}}{p+1}]' \end{aligned}$$

for $M(1)$,

$$h = [r, \frac{r^2}{2}, \cdots, \frac{r^k}{k}, \frac{r^{k+1}}{k+1}, \frac{r^{k+2}}{k+2} - [r^{k+1}(\frac{r}{k+2} - \frac{\lambda}{k+1}) - \lambda^{k+2}(\frac{1}{k+2}$$

$$-\frac{1}{k+1})]\iota_2, [r^{k+1}(\frac{r}{k+2} - \frac{\lambda}{k+1}) - \lambda^{k+2}(\frac{1}{k+2} - \frac{1}{k+1})]\iota_2,$$
$$\cdots, \frac{r^{\ell+1}}{\ell+1} - [r^{k+1}(\frac{r^{\ell-k}}{\ell+1} - \frac{\lambda^{\ell-k}}{k+1}) - \lambda^{\ell+1}(\frac{1}{\ell+1} - \frac{1}{k+1})]\iota_2,$$
$$[r^{k+1}(\frac{r^{\ell-k}}{\ell+1} - \frac{\lambda^{\ell-k}}{k+1}) - \lambda^{\ell+1}(\frac{1}{\ell+1} - \frac{1}{k+1})]\iota_2, \frac{r^{\ell+2}}{\ell+2}, \cdots, \frac{r^{p+1}}{p+1}]'$$

for $M(2)$.

Remark: $h(r)$ is given by

$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2]'$ for Model 1,

$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2, \frac{1}{2}r^2]$ for Model 2,

$[r, \frac{1}{2}r^2\iota_1 + \frac{1}{2}(r^2 - 2\lambda r + \lambda^2)\iota_2, \frac{1}{2}(r^2 - 2\lambda r + \lambda^2)\iota_2]'$ for Model 3,

$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2, \frac{1}{2}r^2\iota_1 + \frac{1}{2}\lambda^2\iota_2, \frac{1}{2}(r^2 - \lambda^2)\iota_2]'$ for Model 4.

Suppose that there are q structural breaks at $T_i = \lambda_i T$ for $\lambda_i \in (0,1)$, $i = 1,\cdots,q$. Again, partial and pure structural breaks are allowed. Without loss of generality, assume $0 < \lambda_1 < \cdots < \lambda_q < 1$. Then, for $M(1)$ and $M(2)$, d_t and f can be written as follows:

$$d_t = [1, t, \cdots, t^{k-1}, t^k\iota_1, \cdots, t^k\iota_q, \cdots, t^\ell\iota_1, \cdots, t^\ell\iota_{q+1}, t^{\ell+1}, \cdots, t^p]'$$

$$f(r) = [1, r, \cdots, r^{k-1}, r^k\iota_1, \cdots, r^k\iota_q, \cdots, r^\ell\iota_1, \cdots, r^\ell\iota_{q+1}, r^{\ell+1}, \cdots, r^p]'$$

and

$$\begin{aligned} d_t \quad = \quad & [1, t, \cdots, t^{k-1}, t^k, t^k(t\eta_1 - (t-T_1)\eta_2), t^k((t-T_1)\eta_2 - (t-T_2)\eta_3), \\ & \cdots, t^k(t-T_q)\eta_{q+1}, \cdots, t^k(t^{\ell-k}\eta_1 - (t^{\ell-k} - T_1^{\ell-k})\eta_2, \\ & \cdots, t^k(t^{\ell-k} - T_q^{\ell-k})\eta_{q+1}, t^{\ell+1}, \cdots, t^p]' \end{aligned}$$

$$
\begin{aligned}
f(r) \;=\;& [1, r, \cdots, r^{k-1}, r^k, r^k(r\eta_1 - (r-\lambda_1)\eta_2), r^k((r-\lambda_1)\eta_2 - (r-\lambda_2)\eta_3), \\
& \cdots, r^k(r-\lambda_q)\eta_{q+1}, \cdots, r^k(r^{\ell-k}\eta_1 - (r^{\ell-k} - \lambda_1^{\ell-k})\eta_2, \\
& \cdots, r^k(r^{\ell-k} - \lambda_q^{\ell-k})\eta_{q+1}, r^{\ell+1}, \cdots, r^p]',
\end{aligned}
$$

where $\eta_i = \sum_{j=i}^{q+1} \iota_j$ and ι_i is an indicator function such that $\iota_i = 1$ if $T\lambda_{i-1} < t \leq T\lambda_i$ for $i = 1,\cdots,q+1$, $\lambda_i \in (0,1)$, $T\lambda_0 = 1$ and $T\lambda_{q+1} = T$.

These specification allow us to apply stationarity tests for multiple time series with a different number of structural breaks for different series when we know the maximum number of structural breaks. This is because the parameters and residuals were consistently estimated when the number of breaks for each series is less than the number specified. Clearly, when $k = 0$ and $\ell = p = q = 1$, the model reduces to Model 3 or Model 4. The asymptotic distributions are obtained straightforwardly, and are reported in the following lemma.

Theorem 5. *Under the same conditions in Theorem 3 with multiple structural breaks, the results of Theorem 3 hold with proper replacement of* $f(r)$ *and* $h(r) = \int_0^r f(s)\, ds$.

5 Asymptotic Distribution with Unknown Break Points

The main criticisms against the results in Section 4 is that the structural break points are assumed to be known. To allow for unknown changing points, we will take the supremum over the range of λ in the manner of Zivot and Andrew [32]. Define $\sup Q_{iT}$ by taking the supremum of these test statistics over λ in a sample of size T. Here, Q_{iT} denotes LM_I, LM_{II}, $SBDH_I$, and $SBDH_{II}$, and Q_i its limiting distribution, for i =1,2,3 and 4, respectively. The limiting distributions are reported in the following theorem.

Theorem 6. *Suppose assumptions A1 - A9 hold.*

(a) Under the null hypothesis with q structural breaks at unknown points $T_i = \lambda_i T,\ \lambda_i \in (0,1)$, $i = 1, \cdots, q$,

$$supQ_{kT} \Rightarrow \sup_{\lambda\in(0,1)} (Q_k),\ \text{for}\ \ k = 1,2,3,4.$$

(b) Under the alternative hypothesis,

$$supQ_{kT} = O_p(T^{2(1-\delta)}),\ k = 1,$$

$$supQ_{kT} = O_p(T^{1-\delta}),\ k = 2,3,4.$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) The above asymptotic distribution could be used for a properly demeaned and/or detrended series with structural breaks. When there is no structural break, one can use the results from Choi and Ahn [9].

(b) The simulated percentiles for Models 1 to 4 for $n = 1, \cdots, 5$ with $q = 1$ are reported in Table 1-4. The results are obtained by taking $\lambda \in (0.15, 0.85)$ with interval 0.02.

6 Finite Sample Power

In this section, we use simulation to investigate the finite sample performance of the tests introduced in Sections 5. In particular, we compare the testing strategy of applying univariate tests several times to each component of a multiple time series with that of applying multivariate tests to the series. The finite sample size and power of the tests proposed in Section 5 depend on the sample size T, the lag length l for long-run variance estimation, the lag window chosen, and the parameters associated with the DGP of $\{x_t\}$ (see Schmidt and Phillips [28] for related analyses). But the finite sample power and size are invariant to $\boldsymbol{\Sigma}$ because the tests are invariant to non singular transformation and to structural break parameter λ because we take supremum of the tests over the possible range of λ. Further, the finite sample size depends on the initial variable x_0, but the finite sample power of the tests is invariant to x_0. In this section, however, we have used only the Quadratic spectral lag window and have chosen $x_0 = 0$ for all the experimental results. The univariate and multivariate tests are expected to reject too often under the null as the initial variable takes larger values (cf. Choi [7]).

Random numbers for the simulation results were generated by the GAUSS subroutine RNDN. Empirical power was calculated out of 2,000 iterations at $T = 100$, 200 and 400 by using the critical values reported in Table 4.3. The lag length is selected by Andrews' [2] method with $AR(4)$ and $VAR(4)$ approximations for univariate and multivariate series, respectively. In order to make the tests consistent, we impose the restriction that $\hat{l} = 2$ if $\hat{l} \geq T^{\epsilon}$, where $\epsilon = 0.65$.

In Table 5, we report the empirical power of LM_I, LM_{II}, $SBDH_I$ and $SBDH_{II}$ for Model 1 to Model 4. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

Each component of the bivariate time series $\{x_t\}$ is $I(1)$; $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests depends on the initial variable x_0. We investigated the finite sample properties in two ways. First, a null of $I(0)$ for each series at 5% significance level was tested. Second, multivariate tests and the double application of univariate tests are compared. M signifies the model considered. The results for the univariate tests in Part (b) were obtained by calculating the fraction of replications for which the null of $I(0)$ is rejected for at least one series at the 5% level.

Because the nominal frequency of non-rejection for the bivariate series is $0.95^2 = 0.9025$, the numbers for the univariate tests should be compared to $1 - 0.9025 \simeq 0.1$. When the numbers are greater than 0.1, the univariate tests are thought to reject too often under the null. For meaningful comparisons, we calculated the fraction of replications for which the multivariate tests reject the null at the 10% level. In Part (a), the results for the tests on each series are reported. M indicates the model used to test the null hypothesis. Consider univariate tests of Model 1. $SBDH_I$ and $SBDH_{II}$ are the most powerful tests for all cases. LM_I is powerful at $T = 400$. LM_{II} is the least powerful in all cases. Comparing univariate and multivariate tests, univariate $SBDH_I$ and $SBDH_{II}$ are slightly more powerful than their multivariate counterparts at $T = 100$ and 200, and are equally powerful at $T = 400$. However, multivariate LM_I and LM_{II} are more powerful than their univariate counterparts. Among the multivariate tests, LM_{II} is least powerful. However, at $T = 400$, power increases significantly.

Across models, our general conclusions are still valid. However, it should be noted that the power decreases as we move from Model 1 to Model 4, although at T =400, the $SBDH$ tests become equally powerful across different models. In Part (b), the multivariate tests are less sensitive to the choice of model than are their univariate counterparts.

In Table 6, we report the empirical power of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

Note that $x_t^{(1)} = I(1)$, $x_t^{(2)} = I(0)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests does not depend on the initial variable x_0. In Part (a), we report the power and size of the univariate tests for various models. For $x_t^{(1)}$, univariate tests are more powerful than those from the DGP 1. It seems that the additional serial correlation of the error contributes to power performance.

For $x_t^{(2)}$, size distortion is observed at $T = 100$ and 200. At $T = 400$, $SBDH_I$ and $SBDH_{II}$ maintain the nominal significance level relatively well. However, LM_I and LM_{II} do not reject the null frequently enough.

In Part (*b*), univariate $SBDH_I$ and $SBDH_{II}$ are slightly more powerful than their multivariate counterparts. Multivariate LM_{II} is equally as powerful as $SBDH_I$ and $SBDH_{II}$ at $T = 200$ and 400. Across models, it is observed that univariate tests for $x_t^{(1)}$ become less powerful as we depart from Model 1. For $x_t^{(2)}$, size distortion increases as we deviate from Model 1. Multivariate tests also become less powerful as we move away from Model 1. However, multivariate tests are less sensitive to the choice of models than are their univariate counterparts.

In Table 7, we report the empirical size of LM_I, LM_{II}, $SBDH_I$ and $SBDH_{II}$ for the data generated by

$$x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

Note that $x_t^{(1)}$, $x_t^{(2)} = I(0)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. Note that the size of all the tests depends on the initial variable x_0 in finite samples. In Part (a), we report the size of the univariate tests. As in Table 6, the univariate tests suffer size distortions at $T = 100$ and $T = 200$. Both $SBDH_I$ and $SBDH_{II}$ keep their nominal size at $T = 400$. Further, it is observed that the size distortion increases when an MA component is included. Again, LM_I and LM_{II} do not reject the null frequently enough. In Part (*b*), size distortion is observed. The size distortion, however, is smaller in multivariate tests than in their univariate counterparts. When $T = 400$, multivariate tests maintain nominal size reasonably well, and $SBDH_I$ and LM_{II} reject the null slightly less than the 10% level. Across models, size distortion is qualitatively the same. However, it disappears in large samples.

The univariate tests are shown to reject the null less frequently than their multivariate counterparts except in the case of $SBDH_{II}$. However, both sets of tests show serious size distortions at $T = 100$. At T =400, though, the multivariate tests have empirical size reasonably close to 0.1, except in the case of LM_{II}. Comparing the four tests, LM_I, $SBDH_I$ and $SBDH_{II}$ tend to reject more often than LM_{II} in all cases. The results reported in Part $(b) - (d)$ for Models 2 - 4 are similar to Part (a).

To summarize our findings

(i) Multivariate LM_{II} suffers size distortion in a negative direction. When $T = 200, 400$, LM_{II} become significantly powerful. Multivariate LM_I keep their nominal size at $T = 400$ and are also powerful.

(ii) For all models considered, the multivariate tests maintain their nominal size well relative to their univariate counterparts.

(iii) For all models, all tests suffer from size distortions at sample sizes $T = 100$ and 200.

7 Summary and Further Remarks

In this paper, we have introduced tests for the null of stationarity in the presence of structural breaks against the alternative of nonstationarity. These tests are applicable to univariate as well as multiple time series for which tests are not available currently. The asymptotic distributions were obtained in a unified manner by using the standard vector Brownian motion and test consistency was established. The effects of omitted structural breaks were analyzed. Simulation results indicate that the tests we have introduced work reasonably well in finite samples and that using the multivariate tests is a better testing strategy than applying the univariate tests several times to each component of a multiple time series. Among the multivariate tests we introduced, the LM_I, $SBDH_I$ and $SBDH_{II}$ tests show the best performance and are recommended for empirical work.

Table 1 Empirical Percentiles for Sup Tests (Model 1)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	10.2701	0.1195	0.2938
	0.850	0.2500	11.2674	0.1356	0.3289
	0.900	0.2500	12.4935	0.1588	0.3770
	0.950	0.2500	14.4704	0.1987	0.4646
	0.975	0.2500	16.3858	0.2424	0.5526
	0.990	0.2500	18.8052	0.2982	0.6690
n = 2	0.800	0.7005	21.8698	0.2196	0.4813
	0.850	0.7550	23.1178	0.2409	0.5253
	0.900	0.8375	24.6907	0.2711	0.5873
	0.950	0.9978	27.2272	0.3215	0.6891
	0.975	1.1763	29.6819	0.3740	0.7892
	0.990	1.4634	32.5821	0.4398	0.9181
n = 3	0.800	1.2970	36.7900	0.3134	0.6471
	0.850	1.4014	38.3025	0.3393	0.6985
	0.900	1.5492	40.2897	0.3742	0.7661
	0.950	1.8291	43.2363	0.4312	0.8802
	0.975	2.1340	46.0330	0.4855	0.9887
	0.990	2.5669	49.1847	0.5595	1.1346
n = 4	0.800	2.0233	55.3425	0.4054	0.8067
	0.850	2.1717	57.1543	0.4345	0.8632
	0.900	2.3851	59.5519	0.4738	0.9400
	0.950	2.7735	63.1246	0.5357	1.0625
	0.975	3.1798	66.3569	0.5938	1.1768
	0.990	3.7945	70.1290	0.6742	1.3162
n = 5	0.800	2.8821	77.5691	0.4954	0.9619
	0.850	3.0875	79.5690	0.5276	1.0230
	0.900	3.3765	82.2933	0.5701	1.1034
	0.950	3.8949	86.2202	0.6384	1.2404
	0.975	4.4437	89.7318	0.6994	1.3682
	0.990	5.1828	94.3984	0.7837	1.5153

Table 2 Empirical Percentiles for Sup Tests (Model 2)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	14.5149	0.0712	0.1159
	0.850	0.2500	15.5144	0.0794	0.1275
	0.900	0.2500	16.9126	0.0909	0.1436
	0.950	0.2500	19.1359	0.1107	0.1716
	0.975	0.2500	21.0785	0.1316	0.2001
	0.990	0.2500	23.6560	0.1575	0.2391
n = 2	0.800	0.6632	27.8936	0.1322	0.1956
	0.850	0.7019	29.2155	0.1427	0.2102
	0.900	0.7589	30.8900	0.1573	0.2298
	0.950	0.8670	33.5073	0.1818	0.2639
	0.975	0.9857	35.8807	0.2059	0.2958
	0.990	1.1709	38.7996	0.2377	0.3388
n = 3	0.800	1.1553	44.8072	0.1869	0.2696
	0.850	1.2223	46.3811	0.1987	0.2863
	0.900	1.3129	48.4144	0.2156	0.3085
	0.950	1.4773	51.6770	0.2432	0.3455
	0.975	1.6463	54.5468	0.2698	0.3835
	0.990	1.8884	58.0719	0.3061	0.4302
n = 4	0.800	1.7175	65.2120	0.2394	0.3412
	0.850	1.8071	67.0840	0.2523	0.3596
	0.900	1.9336	69.3687	0.2707	0.3846
	0.950	2.1577	73.1376	0.3004	0.4251
	0.975	2.3849	76.4737	0.3290	0.4643
	0.990	2.6885	80.2510	0.3667	0.5190
n = 5	0.800	2.3551	89.1968	0.2924	0.4097
	0.850	2.4744	91.4069	0.3066	0.4299
	0.900	2.6351	94.0622	0.3262	0.4572
	0.950	2.9252	98.4637	0.3575	0.5018
	0.975	3.2159	102.2493	0.3873	0.5430
	0.990	3.5917	107.0198	0.4267	0.5963

Table 3 Empirical Percentiles for Sup Tests (Model 3)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	12.6307	0.0645	0.0900
	0.850	0.2500	13.6553	0.0713	0.1002
	0.900	0.2500	14.9880	0.0809	0.1145
	0.950	0.2500	17.1639	0.0970	0.1386
	0.975	0.2500	19.3133	0.1141	0.1637
	0.990	0.2500	22.0483	0.1368	0.1960
n = 2	0.800	0.6091	26.4891	0.1181	0.1608
	0.850	0.6424	27.7868	0.1269	0.1736
	0.900	0.6932	29.5474	0.1391	0.1912
	0.950	0.7937	32.4234	0.1592	0.2199
	0.975	0.9028	35.0448	0.1789	0.2490
	0.990	1.0736	38.0954	0.2044	0.2878
n = 3	0.800	1.0699	43.4910	0.1693	0.2278
	0.850	1.1315	45.1050	0.1793	0.2425
	0.900	1.2196	47.2527	0.1938	0.2626
	0.950	1.3799	50.5990	0.2173	0.2951
	0.975	1.5589	53.6760	0.2386	0.3268
	0.990	1.7992	57.3629	0.2692	0.3687
n = 4	0.800	1.6118	63.9991	0.2196	0.2929
	0.850	1.7002	65.9334	0.2314	0.3093
	0.900	1.8287	68.4353	0.2473	0.3310
	0.950	2.0475	72.4165	0.2721	0.3662
	0.975	2.2823	76.0654	0.2951	0.4009
	0.990	2.6039	80.2543	0.3229	0.4425
n = 5	0.800	2.2323	88.1603	0.2691	0.3556
	0.850	2.3514	90.3943	0.2818	0.3732
	0.900	2.5217	93.3097	0.2977	0.3970
	0.950	2.8129	97.7634	0.3255	0.4363
	0.975	3.1130	101.7885	0.3513	0.4735
	0.990	3.5512	106.6342	0.3849	0.5203

Table 4 Empirical Percentiles for Sup Tests (Model 4)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	16.9971	0.0587	0.0932
	0.850	0.2500	18.0852	0.0647	0.1023
	0.900	0.2500	19.6210	0.0728	0.1156
	0.950	0.2500	21.9213	0.0872	0.1382
	0.975	0.2500	24.0336	0.1027	0.1602
	0.990	0.2500	26.8095	0.1224	0.1893
n = 2	0.800	0.7046	31.5807	0.1388	0.1589
	0.850	0.7521	32.9476	0.1488	0.1706
	0.900	0.8224	34.7221	0.1625	0.1868
	0.950	0.9572	37.5606	0.1840	0.2135
	0.975	1.0991	40.0479	0.2032	0.2401
	0.990	1.3269	43.0335	0.2290	0.2718
n = 3	0.800	1.2506	50.6911	0.1966	0.2204
	0.850	1.3324	52.2817	0.2084	0.2339
	0.900	1.4511	54.4116	0.2234	0.2514
	0.950	1.6736	57.6043	0.2479	0.2816
	0.975	1.8979	60.5734	0.2689	0.3106
	0.990	2.2685	64.0251	0.2953	0.3487
n = 4	0.800	1.8869	72.8866	0.2515	0.2800
	0.850	2.0066	74.8994	0.2636	0.2951
	0.900	2.1806	77.4186	0.2790	0.3158
	0.950	2.4818	81.3086	0.3048	0.3487
	0.975	2.8055	84.9986	0.3271	0.3778
	0.990	3.2935	89.4487	0.3575	0.4182
n = 5	0.800	2.6328	98.9963	0.3054	0.3401
	0.850	2.7935	101.2260	0.3182	0.3562
	0.900	3.0171	104.0339	0.3350	0.3782
	0.950	3.4283	108.3464	0.3611	0.4128
	0.975	3.8511	112.2954	0.3866	0.4442
	0.990	4.4267	117.1247	0.4144	0.4862

Table 5 Empirical Power of Sup Tests

$$DGP\ 1 : x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.34	0.00	0.93	0.91	0.29	0.00	0.89	0.85
2	0.14	0.00	0.83	0.79	0.10	0.00	0.77	0.72
3	0.00	0.00	0.77	0.72	0.00	0.00	0.70	0.65
4	0.00	0.00	0.77	0.78	0.00	0.00	0.69	0.69
	T = 200							
1	0.67	0.02	0.98	0.97	0.64	0.02	0.97	0.95
2	0.60	0.00	0.93	0.89	0.55	0.00	0.87	0.80
3	0.31	0.00	0.92	0.87	0.27	0.00	0.86	0.76
4	0.23	0.00	0.90	0.87	0.19	0.00	0.81	0.76
	T = 400							
1	0.89	0.25	1.00	1.00	0.88	0.24	0.99	0.99
2	0.87	0.08	0.99	0.98	0.08	0.07	0.95	0.93
3	0.69	0.03	0.98	0.97	0.66	0.02	0.95	0.92
4	0.60	0.01	0.98	0.97	0.55	0.00	0.95	0.91
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.38	0.00	0.96	0.94	0.88	0.19	0.93	0.90
2	0.17	0.00	0.92	0.89	0.65	0.04	0.85	0.79
3	0.00	0.00	0.87	0.84	0.59	0.05	0.79	0.75
4	0.00	0.00	0.89	0.89	0.44	0.02	0.70	0.80
	T = 200							
1	0.71	0.02	0.99	0.98	0.97	0.62	0.98	0.97
2	0.65	0.00	0.95	0.92	0.90	0.35	0.93	0.91
3	0.35	0.00	0.94	0.91	0.85	0.33	0.91	0.88
4	0.26	0.00	0.94	0.91	0.82	0.29	0.89	0.90
	T = 400							
1	0.91	0.27	1.00	1.00	1.00	0.90	1.00	1.00
2	0.89	0.09	0.99	0.98	0.98	0.78	0.98	0.97
3	0.72	0.03	0.99	0.98	0.97	0.65	0.98	0.97
4	0.63	0.01	0.99	0.98	0.97	0.72	0.98	0.97

Table 6 Empirical Power of Sup Tests

$$DGP\ 2: x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0,\Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.42	0.00	0.96	0.95	0.01	0.00	0.28	0.20
2	0.22	0.00	0.91	0.88	0.00	0.00	0.34	0.28
3	0.01	0.00	0.84	0.82	0.00	0.00	0.30	0.23
4	0.00	0.00	0.86	0.89	0.00	0.00	0.36	0.30
	T = 200							
1	0.71	0.03	0.99	0.99	0.00	0.00	0.07	0.06
2	0.68	0.00	0.97	0.95	0.00	0.00	0.13	0.10
3	0.38	0.00	0.94	0.91	0.00	0.00	0.11	0.11
4	0.28	0.00	0.96	0.94	0.00	0.00	0.14	0.14
	T = 400							
1	0.89	0.27	1.00	1.00	0.00	0.00	0.02	0.07
2	0.90	0.10	0.99	0.99	0.00	0.00	0.06	0.06
3	0.73	0.04	0.99	0.99	0.00	0.00	0.04	0.07
4	0.64	0.01	0.99	0.99	0.00	0.00	0.06	0.08
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.43	0.00	0.96	0.96	0.94	0.52	0.94	0.92
2	0.23	0.00	0.93	0.91	0.82	0.20	0.89	0.82
3	0.01	0.00	0.88	0.85	0.75	0.30	0.83	0.72
4	0.00	0.00	0.91	0.91	0.66	0.16	0.75	0.82
	T = 200							
1	0.71	0.03	0.99	0.99	0.97	0.91	0.96	0.95
2	0.68	0.00	0.97	0.95	0.89	0.74	0.89	0.85
3	0.38	0.00	0.94	0.92	0.87	0.76	0.85	0.79
4	0.28	0.00	0.96	0.95	0.85	0.67	0.84	0.84
	T = 400							
1	0.89	0.27	1.00	1.00	1.00	0.99	0.99	0.99
2	0.90	0.10	0.99	0.99	0.98	0.96	0.97	0.96
3	0.73	0.04	0.99	0.99	0.97	0.96	0.96	0.94
4	0.64	0.01	0.99	0.99	0.97	0.96	0.96	0.95

Table 7 Empirical Size of Sup Tests

$$DGP\ 3: x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.01	0.00	0.27	0.18	0.03	0.00	0.45	0.35
2	0.00	0.00	0.34	0.28	0.01	0.00	0.47	0.40
3	0.00	0.00	0.30	0.25	0.00	0.00	0.40	0.32
4	0.00	0.00	0.35	0.30	0.00	0.00	0.48	0.42
	T = 200							
1	0.00	0.00	0.08	0.06	0.02	0.00	0.20	0.12
2	0.00	0.00	0.13	0.10	0.02	0.00	0.19	0.14
3	0.00	0.00	0.09	0.10	0.01	0.00	0.19	0.13
4	0.00	0.00	0.14	0.13	0.00	0.00	0.20	0.16
	T = 400							
1	0.00	0.00	0.03	0.06	0.00	0.00	0.03	0.05
2	0.00	0.00	0.07	0.08	0.00	0.00	0.05	0.05
3	0.00	0.00	0.04	0.09	0.00	0.00	0.03	0.06
4	0.00	0.00	0.06	0.09	0.00	0.00	0.04	0.07
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.03	0.00	0.54	0.44	0.38	0.05	0.44	0.32
2	0.01	0.00	0.62	0.54	0.28	0.01	0.47	0.38
3	0.00	0.00	0.54	0.46	0.26	0.01	0.41	0.36
4	0.00	0.00	0.64	0.57	0.19	0.01	0.34	0.45
	T = 200							
1	0.02	0.00	0.24	0.17	0.23	0.04	0.19	0.13
2	0.02	0.00	0.27	0.21	0.20	0.02	0.23	0.19
3	0.01	0.00	0.24	0.20	0.21	0.01	0.22	0.21
4	0.00	0.00	0.30	0.25	0.17	0.01	0.14	0.26
	T = 400							
1	0.00	0.00	0.05	0.09	0.12	0.01	0.04	0.09
2	0.00	0.00	0.10	0.11	0.11	0.01	0.10	0.12
3	0.00	0.00	0.06	0.12	0.13	0.01	0.08	0.15
4	0.00	0.00	0.09	0.13	0.08	0.01	0.02	0.17

APPENDIX: PROOFS

For the proof in this section, we will use the weak convergence results in Phillips and Durlauf [25], Park and Phillips [22] and Chan and Wei [6] freely without refering to these articles in each instances.

Proof of Theorem 1.

The true DGP is given by

$$\begin{aligned} y_t &= A_1 c_t \iota_1 + A_2 c_t \iota_2 + x_t, \\ &= A_1 c_t + (A_2 - A_1) c_t \iota_2 + x_t, \end{aligned} \tag{13}$$

whereas a researcher runs the following regression equation to estimate the residuals:

$$y_t = A_1 c_t + x_t. \tag{14}$$

The residual from (14) is given by

$$\begin{aligned} \bar{x}_t &= x_t - \sum x_t c_t' (\sum c_t c_t')^{-1} c_t + (A_2 - A_1) c_t \iota_2 \\ &\quad - (A_2 - A_1) \sum \iota_2 c_t c_t' (\sum c_t c_t')^{-1} c_t \end{aligned}$$

and

$$\begin{aligned} \bar{S}_t &= S_t - \sum x_t c_t' (\sum c_t c_t')^{-1} S_t^c + (A_2 - A_1) S_t^{c\iota_2} \\ &\quad - (A_2 - A_1) \sum \iota_2 c_t c_t' (\sum c_t c_t')^{-1} S_t^c \\ &= E_{1t} - E_{2t} - E_{3t} - E_{4t}, \end{aligned}$$

where $S_t^w = \sum_{i=1}^t w_i$. Note that $\delta_T^{-1}\ c_{[Tr]} \to c(r)$, $T^{-1}\ \delta_T^{-1}\ S_{[Tr]}^c \to \int_0^r c(s)ds$, and $T^{-1}\ \delta_T^{-1}\ S_{[Tr]}^{c\iota_2} \to \int_0^r c\iota_2(s)ds$. Also, $T^{-1/2}\ E_{1[Tr]} \Rightarrow B(r)$, $T^{-1/2}\ E_{2[Tr]} \Rightarrow \int_0^1 dBc'\ (\int_0^1 cc')^{-1} \bar{c}\,(r)$, and $T^{-1/2}\ E_{3[Tr]}$, $T^{-1/2} E_{4[Tr]}$

$= O_p(T^{p+1/2})$. Since $\bar{S}_t$ is dominated by E_3 and E_4 of $O_p(T^{p+1/2})$, letting $E_t = E_{3t} + E_{4t}$, we have

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}_t' &= T^{-2}\sum_{t=1}^{T}E_tE_t' + O_p(T^{p+1/2}) \\ &= O_p(T^{2p+1}) + O_p(T^{p+1/2}). \end{aligned} \tag{15}$$

Further, by KPSS [17] and Choi and Ahn [9], the long run covariance matrix estimate is given by

$$\begin{aligned} \bar{\Omega}_l &= \sum_{h=-l}^{l}\bar{C}(h)k(h/l) \\ &= O_p(T^{2p+\delta}). \end{aligned} \tag{16}$$

From (15) and (16) the result (iv) follows.

As for (i) to (iii), the following equation is used to formulate test statistics:

$$P_t = A_1 g_t + S_t,$$

where $P_t = \sum_{j=1}^{t} y_t$ and $g_t = \sum_{j=1}^{t} c_j$. The residual is given by

$$\begin{aligned} \tilde{S}_t &= S_t - \sum S_t g_t' \left(\sum g_t g_t'\right)^{-1} g_t \\ &\quad + (A_2 - A_1)\, S_t^{c\iota_2} - (A_2 - A_1)\sum S_t^{c\iota_2} g_t' \left(\sum g_t g_t'\right)^{-1} g_t \\ &= F_{1t} - F_{2t} + F_{3t} - F_{4t} \end{aligned}$$

Note that $\frac{1}{T}\delta_T^{-1} g_{[Tr]} \rightarrow g(r)$, and $\frac{1}{T}\delta_T^{-1}\ S_{[Tr]}^{c\iota_2} \rightarrow \int_0^r c\iota_2(s)ds$. Also, $\frac{1}{\sqrt{T}}F_{1[Tr]} \Rightarrow B(r)$, $\frac{1}{\sqrt{T}}F_{2[Tr]} \Rightarrow \int_0^1 dBf'(\int_0^1 gg')^{-1}g(r)$, and $\frac{1}{\sqrt{T}}F_{3[Tr]}$, $\frac{1}{\sqrt{T}}F_{4[Tr]} = O_p(T^{p+1/2})$. Since $\tilde{S}_t$ is dominated by F_3 and F_4 of $O_p(T^{p+1/2})$, letting $F_t = F_{3t} + F_{4t}$, we have:

$$\frac{1}{T^2}\sum_{t=1}^{T}\tilde{S}_t\tilde{S}_t' = \frac{1}{T^2}\sum_{t=1}^{T}F_tF_t' + O_p\left(T^{p+1/2}\right) = O_p\left(T^{2p+1}\right)$$

and

$$\frac{1}{T}\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}_{t-1}' = O_p\left(T^{2p+1}\right).$$

Also, by KPSS [17] and Choi and Ahn [9], the long run covariance matrix estimate is given as

$$\tilde{\Omega}_l = \sum_{h=-l}^{l}\tilde{C}(h)\,k\left(\frac{h}{l}\right) = O_p\left(T^{2p+\delta}\right).$$

(i) to (iii) follow immediately from the above result. For further details, see Choi and Ahn [9].

Proof of Theorem 2.

Part (a): The true DGP is represented by

$$y_t = Ad_t + x_t \ \text{ for the bar case,} \tag{17}$$

and

$$S_t^y = Ah_t + S_t \ \text{ for the tilde case.} \tag{18}$$

Define the weight matrix

$$\delta_T = diag[T^0, T, \cdots, T^{k-1}, T^k, T^k, \cdots, T^\ell, T^\ell, T^{\ell+1}, \cdots, T^p].$$

Then, $\delta_T^{-1}d_{[Tr]} \to f(r)$ and $T^{-1}\delta_T^{-1}h_{[Tr]} \to h(r) = \int_0^r f(s)ds$. The limiting distributions for the OLS estimators from equations (8) and (11) are given by

$$\begin{aligned} T^{1/2}(\bar{A}-A)\delta_T &\Rightarrow \int_0^1 dBf'(\int_0^1 ff')^{-1} \\ &= \bar{F}, \end{aligned} \tag{19}$$

$$\begin{aligned} T^{1/2}(\tilde{A}-A)\delta_T &\Rightarrow \int_0^1 Bh'(\int_0^1 hh')^{-1} \\ &= \tilde{F}. \end{aligned} \tag{20}$$

Now consider the moment matrix of OLS residuals $\bar{S}_t = \sum_{i=1}^{t} \bar{x}_t$ and $\tilde{S}_t$.

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}_t' &= T^{-2}\sum_{t=1}^{T}(S_t-(\bar{A}-A)h_t)(S_t-(\bar{A}-A)h_t)' \\ &\Rightarrow \int_0^1 (B-\bar{F}h)(B-\bar{F}h)', \qquad (21) \\ T^{-2}\sum_{t=1}^{T}\tilde{S}_t\tilde{S}_t' &= T^{-2}\sum_{t=1}^{T}(S_t-(\tilde{A}-A)h_t)(S_t-(\tilde{A}-A)h_t)' \\ &\Rightarrow \int_0^1 (B-\tilde{F}h)(B-\tilde{F}h)', \qquad (22) \end{aligned}$$

and

$$\begin{aligned} T^{-1}\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}_{t-1}' &= T^{-1}\sum_{t=2}^{T}(\Delta S_t-(\tilde{A}-A)d_t)(S_{t-1}-(\tilde{A}-A)h_{t-1})' \\ &\Rightarrow \int_0^1 d(B-\tilde{F}h)(B-\tilde{F}h)' + \Omega_1'. \qquad (23) \end{aligned}$$

Next, the covariance matrices are consistent (see Choi and Ahn [9] for proof)

$$\bar{\Omega}_l, \tilde{\Omega}_l, \bar{\Omega}_1, \tilde{\Omega}_1 \xrightarrow{p} \Omega_l \text{ and } \Omega_1, \text{ respectively.} \qquad (24)$$

Hence, (i) - (iv) of Theorem 2 follows immediately from (19)-(24).

Part (b) : The proof is a simple application of the proof in Choi and Ahn [9] and the proof of Theorem 1 above.

Proof of Theorem 3 and Proof of Theorem 4.

These results are trivially obtained by applying the methods used in the proof of Theorem 2.

Proof of Theorem 5.

To prove Theorem 5, it is enough to show that we can transform the models to the form of $M(1)$ and $M(2)$ when there are multiple structural breaks. The result follows immediately from Theorem 2. It is trivial to transform the model to $M(1)$ without the continuity restriction. Under the continuity restriction, we have the following q restrictions;

$$a_k^1 = a_k^2 + (a_{k+1}^2 - a_{k+1}^1)T_1 + \cdots + (a_\ell^2 - a_\ell^1)T_1^{\ell-k}$$

$$a_k^2 = a_k^3 + (a_{k+1}^3 - a_{k+1}^2)T_2 + \cdots + (a_\ell^3 - a_\ell^2)T_2^{\ell-k}$$

$$\vdots$$

$$a_k^q = a_k^{q+1} + (a_{k+1}^{q+1} - a_{k+1}^q)T_q + \cdots + (a_\ell^{q+1} - a_\ell^q)T_q^{\ell-k}.$$

a_k^i is expressed as follow:

$$a_k^i = a_k^{q+1} + \sum_{h=k+1}^{\ell} \sum_{j=i}^{q} (a_h^{j+1} - a_h^j)T_j^{h-k}. \tag{25}$$

Substituting a_k^i with the right hand side of (25) for $i = 1, \cdots, q$, into equation (12) and using the indicator functions $\eta_i = \sum_{j=1}^{q+1} \iota_j$, the desired equation results. Note that these q restrictions reduces the number of parameters to be estimated by q for each equation.

Proof of Theorem 6.

The desired results follow immediately by applying the continuous mapping theorem to Theorem 3.

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