

# Bayesian Stochastic Frontier Analysis Using WinBUGS

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## Abstract

Markov chain Monte Carlo (MCMC) methods have become a ubiquitous tool in Bayesian analysis. This paper implements MCMC methods for Bayesian analysis of stochastic frontier models using the WinBUGS package, a freely available software. General code for cross-sectional and panel data are presented and various ways of summarizing posterior inference are discussed. Several examples illustrate that analyses with models of genuine practical interest can be performed straightforwardly and model changes are easily implemented.

Keywords: Efficiency, Markov chain Monte Carlo, Model comparison, Regularity, Software

## 1 Introduction

The use of stochastic frontiers in the analysis of productivity and firm efficiency has become widespread since the seminal papers by Meeusen and van den Broeck (1977) and Aigner, Lovell and Schmidt (1977). More recently, a large amount of interest has been devoted to the use of Bayesian methods for making inference in stochastic frontier models. The latter was introduced by van den Broeck *et al.* (1994), who commented on its particular advantages in this context: exact (small-sample) inference on efficiencies, easy incorporation of prior ideas and restrictions such as regularity conditions and optimal treatment of parameter and model uncertainty. Bayesian methods are now commonplace in this literature, as evidenced by Kim and Schmidt (2000) and recent applications by *e.g.* Kurkalova and Carriquiry (2002) and Ennsfellner, Lewis and Anderson (2004). In addition, five of the 12 papers in a recent special issue of the *Journal of Econometrics* on “Current developments in productivity and efficiency measurement” adopt a Bayesian approach. The complexity of stochastic frontier models makes numerical integration methods inevitable. The most appropriate method in this context is clearly Markov chain Monte Carlo (MCMC), as introduced by Koop, Steel and Osiewalski (1995) and used in virtually all recent Bayesian

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papers in this literature, see *e.g.* Kurkalova and Carriquiry (2002), Tsionas (2002), Huang (2004) and Kumbhakar and Tsionas (2005).

A problem that occurs, especially for applied users that have not yet implemented Bayesian methods in this field, is the availability of reliable and user-friendly software. To our knowledge, there is only one publicly available software, which is described in Arickx *et al.* (1997). However, the latter is based on importance sampling, rather than MCMC, and lacks flexibility in model specification (it basically only implements the simple cross-sectional stochastic frontier model with a restricted choice of efficiency distributions and priors).

Thus, this paper describes the use of a freely available software for the analysis of complex statistical models using MCMC techniques, called WinBUGS, in the context of stochastic frontiers. It turns out that WinBUGS can become a very powerful and flexible tool for Bayesian stochastic frontier analysis, and only requires a relatively small investment on the part of the user. Once the (applied) user understands the logic of model building with WinBUGS, Bayesian analysis is conducted quite easily and many built-in features can be accessed to produce an in-depth and interactive analysis of these models. In addition, execution is quite fast, even of complicated models with large amounts of data and model extensions can easily be accommodated in a modular fashion. The modeller can really concentrate on building and refining an appropriate model<sup>1</sup>, without having to invest large amounts of time in coding up the MCMC analysis and the associated processing of the results. Despite the relative ease of use, we do wish to reiterate the health warning that comes with WinBUGS: “The programs are reasonably easy to use and come with a wide range of examples. There is, however, a need for caution. A knowledge of Bayesian statistics is assumed, including recognition of the potential importance of prior distributions, and MCMC is inherently less robust than analytic statistical methods. There is no in-built protection against misuse.”

We illustrate the use and flexibility of WinBUGS for Bayesian stochastic frontier modelling of the cross-sectional data on electric utility companies used and listed in Greene (1990) and of the panel data on hospitals used in Koop, Osiewalski and Steel (1997).

The WinBUGS software (together with a user manual) can be downloaded (the current fee is zero) from the website

<http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/contents.shtml>.

We have used Version 1.4 in this paper. All WinBUGS code used in this paper, as well as the data on the electricity firms and hospitals is freely available at

[http://www.warwick.ac.uk/go/msteel/steel\\_homepage/software/](http://www.warwick.ac.uk/go/msteel/steel_homepage/software/).

## 2 Stochastic Frontier Model

The basic model relates producers’s costs (or outputs) to a minimum cost (or maximum output) frontier. If a panel of costs has been observed, a simple model regresses the logarithm of cost,  $y_{it}$ , associated with firm  $i$  observed at time  $t$  and producing a certain quantity  $Q_{it}$ , on a set of regressors in  $x_{it}$ , which will be

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<sup>1</sup>This can include models which have not previously appeared in the literature, such is the flexibility of the software.

functions of the logarithm of input prices and  $Q_{it}$  ( $i = 1, \dots, N, t = 1, \dots, T$ ):

$$y_{it} \stackrel{i.i.d.}{\sim} \text{N}(\alpha + x'_{it}\beta + u_{it}, \sigma^2), \quad (1)$$

where  $\text{N}(\mu, \sigma^2)$  denotes a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Inefficiencies  $u_{it}$  model the difference between best-practice and actual cost, and these are assumed to have a one-sided distribution, such as the exponential (as *e.g.* in Meeusen and van den Broeck, 1977). Often we will exploit the panel context by assuming that inefficiencies remain constant over time, *i.e.*  $u_{it} = u_i, t = 1, \dots, T$ , leading to<sup>2</sup>

$$u_i \stackrel{i.i.d.}{\sim} \text{Exp}(\lambda), \quad (2)$$

which denotes an exponential distribution with mean  $1/\lambda$ . The parameters introduced in this model are assigned priors, for example a multivariate normal:

$$\beta \sim \text{N}(0, \Sigma),$$

possibly truncated to reflect regularity conditions (see Subsection 3.3),

$$\sigma^{-2} \sim \text{Ga}(a_0, a_1),$$

a gamma distribution with shape parameter  $a_0$  and mean  $a_0/a_1$ , and

$$\lambda \sim \text{Exp}(-\log r^*),$$

where the latter prior implies that prior median efficiency is equal to  $r^*$ . Minor changes are required for production frontiers.<sup>3</sup> Firm-specific efficiencies are introduced as functions of the inefficiency terms; in particular, the efficiency of firm  $i$  is defined as  $r_i = \exp(-u_i)$ . These are clearly key quantities of interest in practice.

In WinBUGS, models are expressed in code through the distributions of the observations and parameters together with their independence structure. A stochastic frontier model is formulated in Display 1 for a potentially unbalanced panel of observations with time-invariant inefficiencies and a cost frontier. Firstly, the distribution of the log cost in equation (1) is coded. The vector `y` holds the  $K$  observed log costs, and the matrix `data` is a  $K \times (p + 2)$  matrix. Each row holds the observed values for each firm of the  $p$  regressors, the  $p+1$ -th column holds an index for each firm running from 1 to  $N$  and the  $p+2$ -th column holds the time of the observation, which has a maximum value of  $T$  (the latter variable is used in Section 4, where we have a time trend in the frontier and allow for efficiencies to vary over time). The distribution of  $y_{it}$  is encoded using the command `dnorm` for the *stochastic node* `y[k]` which has two arguments representing the mean and the precision (the inverse of the variance). WinBUGS restricts the parameters of the command to be variables and so the mean must be defined as a *logical node* `mu[k]` which can be formed in a standard way. The expression `inprod(beta[], data[k, 1:p])` represents  $x'_{it}\beta$ .

<sup>2</sup>For large  $T$  that assumption might be weakened, as we will discuss in Subsection 4.2.

<sup>3</sup>If we wish to model production frontiers,  $y_{it}$  will be the output produced with a certain quantity of inputs, which will determine the regressors in  $x_{it}$  and the inefficiency term  $u_{it}$  will appear with a negative sign in the mean of  $y_{it}$ .

The inefficiency  $u_i$  is specified to have an exponential distribution with mean  $1/\lambda$  using the command `dexp(lambda)`. A useful feature of WinBUGS is the use of logical nodes to define interesting functions of the parameters in the model. The  $i$ -th firm's efficiency is represented by `eff[i]`. Finally, the prior distributions of each unknown parameter is specified as above.

```
model
{
  for ( k in 1:K ) {
    firm[k] <- data[k, p + 1]
    mu[k] <- alpha + u[firm[k]] + inprod(beta[],data[k,1:p])
    y[k] ~ dnorm(mu[k], prec)
  }

  for ( i in 1:N ) {
    u[i] ~ dexp(lambda)
    eff[i] <- exp(- u[i])
  }

  lambda0 <- -log(rstar)
  lambda ~ dexp(lambda0)

  alpha ~ dnorm(0.0, 1.0E-6)
  for ( i in 1:p ) {
    beta[i] ~ dnorm(0.0, 1.0E-6)
  }
  prec ~ dgamma(0.001, 0.001)
  sigmasq <- 1 / prec
}
```

Display 1: WinBUGS model specification code for the basic cost frontier with panel data

It is very easy to change some of the model assumptions above. For example, we may want to use a different distribution for the inefficiencies, such as the half-normal used in Aigner, Lovell and Schmidt (1977) or the truncated normal of Stevenson (1980)<sup>4</sup>. For the half-normal

$$u_i \sim N^+(0, \lambda^{-1})$$

we simply replace the distribution of `u[i]` by

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<sup>4</sup>The truncated normal and half-normal need a “shared component” which can be downloaded from the WinBUGS development site, currently at <http://homepages.tesco.net/~creeping.death/shared.html>.

```
u[i] ~ djl.dnorm.trunc(0,lambda,0,1000),
```

while the truncated normal distribution

$$u_i \sim N^+(\zeta, \lambda^{-1})$$

also allows the mean of the underlying normal distribution to be estimated and is implemented by

```
u[i] ~ djl.dnorm.trunc(zeta,lambda,0,1000)
```

A general gamma  $Ga(\phi, \lambda)$  distribution as in Greene (1990) would correspond to

```
u[i] ~ dgamma(phi,lambda).
```

Appropriate prior specifications for the parameters need to be included. Various suggestions for prior choices have been made in the literature (*e.g.* in Tsionas, 2000 and Griffin and Steel, 2004).

Once the model code has been loaded, the data must be specified in a special format, which is taken from the statistical package, S-plus. More details are available from the WinBUGS manual. Code for converting data from some other popular packages can be found on the page

<http://www.mrc-bsu.cam.ac.uk/bugs/weblinks/webresource.shtml>

Finally, initial values for the variables being estimated need to be specified. The speed of convergence of the chain is affected by these values. Values with larger posterior density will generally lead to faster convergence. However, in our experience convergence of chains is relatively fast from most plausible choices. Once the model, data and initial values have been entered, WinBUGS creates compiled code to perform an MCMC algorithm for sampling from the posterior distribution. There are several sampling options including multiple chains to aid convergence diagnosis and thinning of the chain to reduce dependence between successive simulated values.

The following sections illustrate the power of WinBUGS to produce useful summary statistics and graphical representations of the posterior distribution with several example datasets. We also show how changes to the model specification can be implemented quite easily, and how to deal with economic regularity conditions and model uncertainty.

### 3 An Old Chestnut: The Electricity Data

The first example analyses  $N = 123$  cross-sectional data from the U.S. electric utility industry in 1970. The data was originally analysed by Christensen and Greene (1976) and subsequently by Greene (1990). Following that analysis, we specify the frontier for  $\ln(\text{Cost}/P_f)$  as

$$\alpha + \beta_1 \ln Q + \beta_2 \ln(P_l/P_f) + \beta_3 \ln(P_k/P_f) + \beta_4 \ln^2 Q \quad (3)$$

where Output ( $Q$ ) is produced with three factors: labour, capital, and fuel and the respective factor prices are  $P_l$ ,  $P_k$  and  $P_f$ .

### 3.1 The standard exponential model

Here we use the model in Section 2, where now we have  $T = 1$ , with exponentially distributed inefficiencies. The chain was run with a burn-in of 20 000 iterations with 200 000 retained draws and a thinning to every 5th draw. WinBUGS has a number of tools for reporting the posterior distribution. A simple summary (Table 1) can be generated showing posterior mean, median and standard deviation with a 95% posterior credible interval. Parameter names are related in an obvious way to the model in Section 2. A fuller picture of the posterior distribution can be provided using the `density` option in the `Sample Monitor Tool` which draws a kernel density estimate of the posterior distribution of any chosen parameter, as in Figure 1.

<b>node</b>	<b>mean</b>	<b>sd</b>	<b>MC error</b>	<b>2.5%</b>	<b>median</b>	<b>97.5%</b>	<b>start</b>	<b>sample</b>
alpha	-7.47	0.3407	0.001049	-8.133	-7.472	-6.793	20001	200000
beta[1]	0.4252	0.04301	1.544E-4	0.34	0.4255	0.5088	20001	200000
beta[2]	0.2501	0.06495	1.513E-4	0.1226	0.2499	0.3776	20001	200000
beta[3]	0.0473	0.06193	1.823E-4	-0.07302	0.04679	0.17	20001	200000
beta[4]	0.02962	0.002843	8.653E-6	0.02407	0.02961	0.03524	20001	200000
lambda	12.23	5.207	0.03901	7.072	10.79	26.75	20001	200000
sigmasq	0.01332	0.003836	2.381E-5	0.007233	0.01281	0.02188	20001	200000

Table 1: WinBUGS output for the electricity data: posterior statistics

<b>node</b>	<b>2.5%</b>	<b>median</b>	<b>97.5%</b>
eff[1]	1	4	90
eff[2]	34	99	123
eff[3]	13	74	121
eff[4]	9	58	120
eff[5]	25	91	122
eff[6]	23	89	122
eff[7]	21	84	122

Table 2: WinBUGS output for the electricity data: rank statistics for the first seven firms in the sample

Often, the quantities of primary interest in stochastic frontier analysis are the efficiencies. Firm-specific efficiencies are immediately generated by the sampler for each firm and their full posterior distributions are readily available, and can be plotted in the same way as in Figure 1. There are various other options for displaying the posterior distribution. For example the `Compare...` menu item brings up the `Comparison Tool` that draws a boxplot (Figure 2) or caterpillar plot of the sampled efficiencies for some chosen firms. A practically interesting function of the firm-specific efficiency measurements is given by their ranks. WinBUGS can automatically compute a sample from their posterior

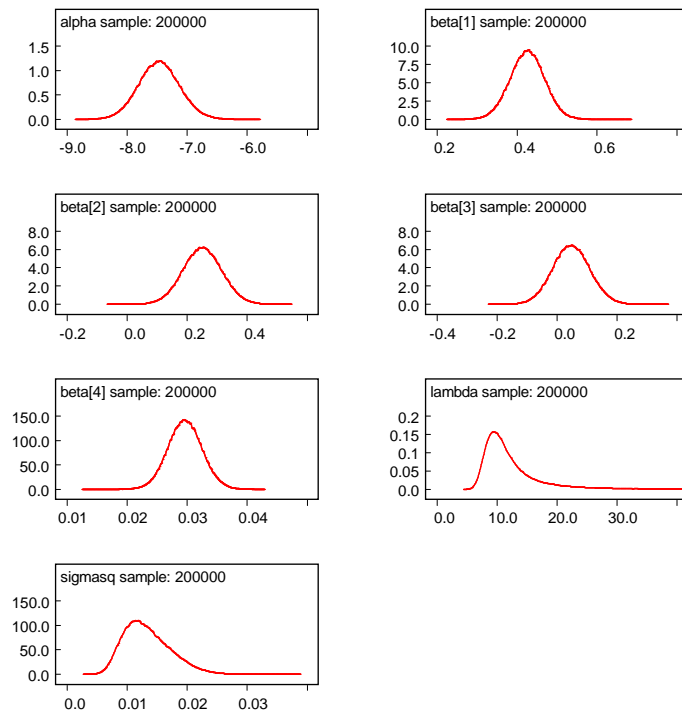


Figure 1: WinBUGS output for the electricity data: posterior densities for parameters

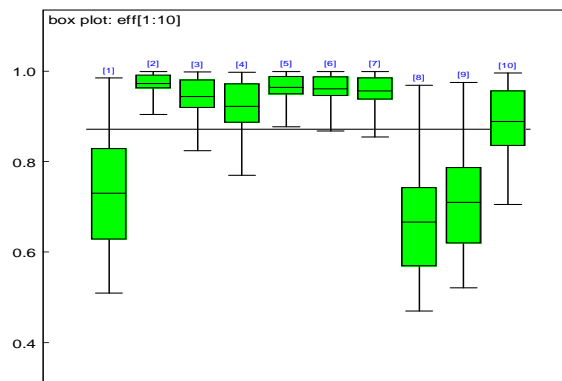


Figure 2: WinBUGS output for the electricity data: boxplot of posterior efficiency distributions for the first ten firms in the sample

distribution using the Rank . . . option from the Inference menu. Table 2 shows a summary of the posterior distribution for the first seven electricity producers in the sample (high rank corresponds to high efficiencies). The posterior distribution clearly demonstrates a large spread of the rankings.

A simple check of the mixing of the posterior distribution arises from some graphical summary of the values taken by the chain. For example, a trace plot of all drawn values is available through the history button or through the trace button for the previous batch of, say, 1000 drawings.

The autocorrelation function for the chain of each parameter (as shown in Figure 3) can also indicate dimensions of the posterior distribution that are mixing slowly. Slow mixing is often associated with high

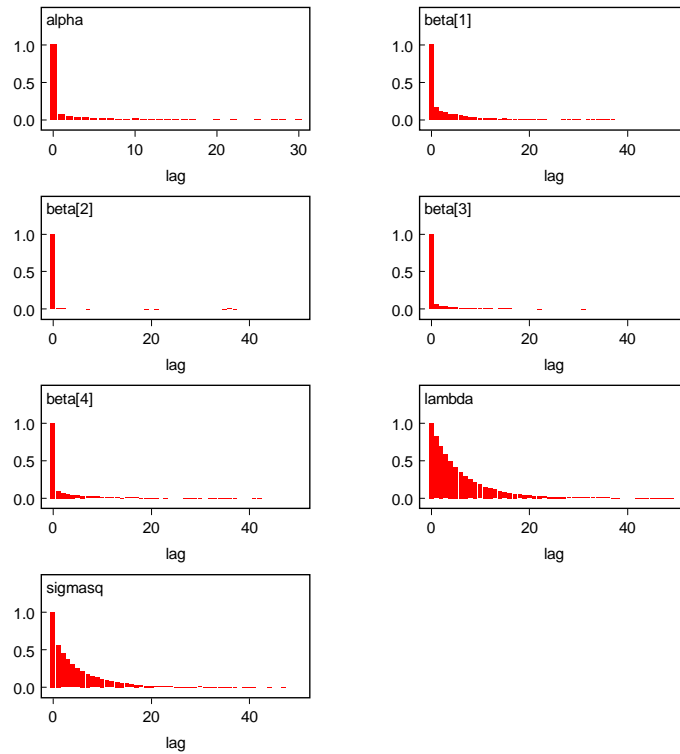


Figure 3: WinBUGS output for the electricity data: autocorrelation functions of the chain

posterior correlations between parameters. The plots indicate that all parameters are mixing well with autocorrelation vanishing before 20 lags in each case. The `correlation` tool (the `correlation` option in the `inference` menu) can produce scatterplots of every parameter against every other parameter to indicate correlation or a correlation coefficient can be estimated from the current output.

Graphical representations of the posterior distribution can indicate problems with the performance of the MCMC algorithm. More sophisticated methods for convergence detection are implemented in the Convergence Diagnostic and Output Analysis (CODA) software which is available for the statistical packages S-plus and R. WinBUGS produces output that is formatted for direct use with these programs and allows the behaviour of the chain to be investigated using some popular statistical tests.

**Dbar = post.mean of  $-2\log L$ ; Dhat =  $-2\log L$  at post.mean of stochastic nodes**

	<b>Dbar</b>	<b>Dhat</b>	<b>pD</b>	<b>DIC</b>
y	-187.486	-233.447	45.961	-141.525
total	-187.486	-233.447	45.961	-141.525

Table 3: WinBUGS output for the electricity data: DIC with normal errors

WinBUGS automatically implements the DIC (Spiegelhalter *et al.*, 2002) model comparison criterion. This is a portable information criterion quantity that trades off goodness-of-fit against a model complexity penalty. In hierarchical models, deciding the model complexity may be difficult and the method estimates the “effective number of parameter”, denoted here by  $p_D$ .  $\bar{D}$  is the posterior mean

of the deviance ( $-2 \times \log$  likelihood) and  $\hat{D}$  is a plug-in estimate of the latter based on the posterior mean of the parameters. The DIC is computed as  $\text{DIC} = \bar{D} + p_D = \hat{D} + 2p_D$ .<sup>5</sup> Lower values of the criterion indicate better fitting models. Table 3 records the values computed, in the same format as given by WinBUGS. For our purposes here, we will focus only on the DIC value. The method was designed to be easy to implement using a sample from the posterior distribution and the interested reader is directed to Spiegelhalter *et al.* (2002) for a lively discussion of its merits.

### 3.2 Alternative distributional assumptions

Once this model has been fitted successfully, we may want to consider further modelling options. As already indicated in Section 2, there are alternative choices of the inefficiency distribution. Another popular choice is the half-normal, which can be implemented in WinBUGS by

```
u[i] ~ djl.dnorm.trunc(0, lambda, 0, 1000).
```

The prior distribution

```
lambda ~ dgamma(1, 1/37.5)
```

leads to a prior median efficiency of approximately 0.875 with a reasonable spread (see van den Broeck *et al.*, 1994). Yet another possible inefficiency distribution is the gamma distribution, implemented by

```
u[i] ~ dgamma(phi, lambda).
```

A suitable prior distribution, which extends the informative prior for an exponential inefficiency distribution, is discussed in Griffin and Steel (2004) in a more general setting. They define

```
d1 <- 3
d2 <- d1 + 1

lambda0 <- -log(rstar)
phi <- 1 / invphi
invphi ~ dgamma(d1, d2)
lambda ~ dgamma(phi, lambda0)
```

The prior on `phi` has mode at one (corresponding to the exponential), so this centres the gamma distribution over the exponential distribution and the parameter `d1` controls the variability of `phi` (a value of 3 is their suggested setting).

Table 4 contrasts some results on the parameters for exponential, half-normal and gamma assumptions. Differences on common parameters are fairly small<sup>6</sup>, and the credible interval for  $\phi$ , the shape parameter of the gamma, includes 1, which corresponds to the exponential model.

In addition, a heavier tailed error distribution could be considered for the measurement error. Simply specifying

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<sup>5</sup>Thus,  $p_D$  is computed as  $\bar{D} - \hat{D}$ .

<sup>6</sup>Note that  $\lambda$  does not share a common interpretation across models.

inefficiency distribution	exponential	half-normal	gamma
$\alpha$	-7.5 (-8.1, -6.8)	-7.4 (-8.0, -6.7)	-7.5 (-8.1, -6.8)
$\beta_1$	0.43 (0.34, 0.51)	0.41 (0.33, 0.48)	0.42 (0.33, 0.50)
$\beta_2$	0.25 (0.12, 0.38)	0.24 (0.11, 0.37)	0.25 (0.12, 0.38)
$\beta_3$	0.05 (-0.07, 0.17)	0.06 (-0.06, 0.19)	0.05 (-0.07, 0.18)
$\beta_4$	0.030 (0.024, 0.035)	0.031 (0.025, 0.036)	0.030 (0.024, 0.036)
$\phi$			1.65 (0.58, 6.15)
$\lambda$	10.8 (7.0, 27.2)	44.5 (20.8, 140.6)	14.5 (6.9, 52.1)
$\sigma^2$	0.013 (0.007, 0.022)	0.013 (0.006, 0.021)	0.013 (0.006, 0.013)

Table 4: Parameter results (posterior medians and 95% credible intervals) for various efficiency distributions with normal errors for the electricity data

$$y[k] \sim dt(\mu[k], \text{prec}, \text{degfree})$$

changes the form of the error distribution in (1) to a  $t$ -distribution with `degfree` degrees of freedom. The prior distribution for the degrees of freedom was chosen to be exponential with mean and standard deviation equal to 3:

$$\text{degfree} \sim \text{dexp}(0.333),$$

which puts a considerable amount of prior mass on distributions with much heavier tails than the normal distribution.

inefficiency distribution	exponential	half-normal	gamma
$\alpha$	-7.8 (-8.4, -7.1)	-7.7 (-8.4, -7.0)	-7.8 (-8.4, -7.1)
$\beta_1$	0.45 (0.36, 0.52)	0.44 (0.35, 0.52)	0.44 (0.35, 0.52)
$\beta_2$	0.29 (0.16, 0.42)	0.28 (0.16, 0.41)	0.29 (0.16, 0.42)
$\beta_3$	0.04 (-0.07, 0.16)	0.05 (-0.06, 0.17)	0.04 (-0.07, 0.16)
$\beta_4$	0.028 (0.023, 0.034)	0.029 (0.023, 0.034)	0.023 (0.028, 0.034)
$\nu$	4.4 (2.0, 11.8)	3.5 (1.6, 10.3)	4.0 (1.7, 11.1)
$\phi$			1.9 (0.7, 6.9)
$\lambda$	11.3 (7.1, 29.4)	43.8 (20.8, 141.1)	16.0 (7.5, 56.1)
$\sigma^2$	0.008 (0.003, 0.016)	0.006 (0.001, 0.014)	0.002 (0.0004, 0.0073)

Table 5: Parameter results (posterior medians and 95% credible intervals) for the two alternative efficiency distributions with  $t$  errors for the electricity data

Table 5 records some results and illustrates that the prior assumption about the degrees of freedom, indicated by  $\nu$ , is quite important since the data provide little information about its value. We can use the DIC criterion to compare the different models. Table 6 compares the DIC scores for the possible combinations of error distribution and inefficiency distribution. Smaller values of the DIC suggest better

error distribution	inefficiency distribution	$\bar{D}$	$\hat{D}$	$p_D$	$DIC$
normal	exponential	-187.5	-233.4	46.0	-141.5
	half-normal	-189.3	-239.5	50.2	-139.1
	gamma	-187.7	-234.0	46.4	-141.3
$t$	exponential	-190.7	-238.8	48.1	-142.6
	half-normal	-209.7	-263.3	53.6	-156.1
	gamma	-198.7	-247.7	49.0	-149.7

Table 6: Comparison of models with different distributional assumptions using the DIC criterion

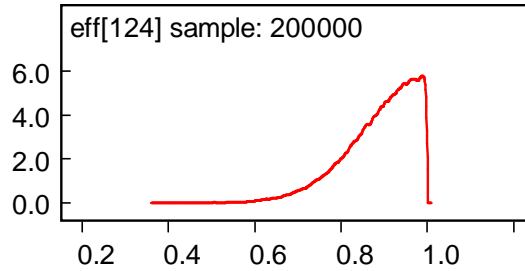


Figure 4: WinBUGS output for the electricity data: kernel density estimate of the posterior predictive efficiency distribution for the Student- $t$  model with half-normal efficiencies

models and so the Student- $t$  errors tend to fit the data better than the normal measurement errors. Overall, the results favour the half-normal distribution with  $t$ -distributed errors. The posterior distribution of the mean of the predictive (*i.e.* out-of-sample) efficiency is a useful measure for comparing our inference about the parameters of the inefficiency distributions  $\lambda$  and  $\phi$  (presented in Table 5). If  $t$ -distributed errors are assumed, the posterior mean of predictive efficiency has a 95% credibility interval of (0.88,0.97) for the exponential distribution, (0.80,0.96) for the gamma and (0.85,0.94) for the half-normal distribution. In other words, for this data, the half-normal and gamma distributions are associated with slightly lower estimates of efficiency than the exponential distribution.

Finally, we estimate the posterior predictive distribution of efficiency for our preferred model with  $t$ -distributed measurement errors and a half-normal efficiency distribution. This corresponds to the efficiency of an unobserved firm in this sector. An extra inefficiency node ( $u[N+1]$ ) and efficiency node ( $eff[N+1]$ ) are added to the model by defining

```

for (i in 1:(N+1)) {
  u[i] ~ djl.dnorm.trunc(0,lambda,0,1000)
  eff[i] <- exp(- u[i])
}

```

A kernel density estimate of the distribution is shown in Figure 4. Posterior predictive median efficiency is 0.91 and the 95% credible interval is (0.69,0.996).

### 3.3 Imposing regularity conditions

The fitted frontier should obey certain economic constraints (see *e.g.* Kumbhakar and Lovell 2000). For example, a cost frontier should imply positive elasticities of cost with respect to output and prices. In other word, we need to check

$$\frac{d \ln C}{d \ln X_i} > 0.$$

If a Cobb-Douglas frontier is fitted the condition reduces to positive coefficients in the frontier. This change can be easily implemented in the prior distribution of  $\beta_i$  by replacing the normal prior

```
beta[i] ~ dnorm(0.0, 1.0E-06)
```

with its truncated counterpart

```
beta[i] ~ djl.dnorm.trunc(0.0, 1.0E-0.6, 0, 1000).
```

However, a more complicated frontier such as a translog will lead to more complicated expressions for  $d \ln C / d \ln X_i$ . In this example, a quadratic output term is included in the frontier and the elasticity of cost with respect to output has the form

$$\frac{d \ln C}{d \ln Q} = \beta_1 + 2\beta_4 \ln Q.$$

Ideally, we would want this relationship to be true for all values of  $\ln Q$ . However, since this is only a local approximation to the frontier, it is usual to check the condition for a plausible set of values of  $\ln Q$ . A pragmatic approach restricts attention to the value of the elasticity for the observed output. This approach will be used in this case. Imposing these properties in alternative sets of values as in Terrell (1996) can easily be implemented. The restrictions are imposed on  $p(y|X, \beta)$  leading to a non-standard likelihood function, which is implemented using the “ones-trick” described in the WinBUGS manual. This introduces a new variable `check`, which will be zero for all cases where regularity conditions are violated and one elsewhere. The data is now a set of ones assumed to be the result of Bernoulli sampling with probabilities proportional to the likelihood values of those observations for which regularity holds and zero for which it is violated. As soon as a violation occurs for one of the observations, the likelihood value associated with that draw will, thus, be zero. `C` is a constant such that all values in `prob` are smaller than one. In particular, we replace

```
for ( k in 1:K ) {  
  firm[k] <- data[k, p + 1]  
  mu[k] <- alpha + u[firm[k]] + inprod(beta[], data[k, 1:p])  
  y[k] ~ dnorm(mu[k], prec)  
}
```

by

```
C<- 10000  
for ( k in 1:K ) {
```

```

ones[k] <- 1
firm[k] <- data[k, p + 1]
mu[k] <- alpha + u[firm[k]] + inprod(beta[1:p], data[k, 1:p])\
      + beta[p + 1] * data[k, 1] * data[k, 1]
check[k] <- step(beta[1] + 2 * beta[4] * data[k, 3])\
      * step(beta[2]) * step(beta[3])
prob[k] <- check[k] * sqrt(prec) * exp(- 0.5 * prec * (y[k] - \
      mu[k]) * (y[k] - mu[k]))/C
ones[k] ~ dbern(prob[k])
}

```

where the more general form of  $\mu$  is used throughout this section and is simply a consequence of the specific cost frontier in (3) and  $\backslash$  indicates that the following line is the continuation of the current line and both should be entered as a single line.

inefficiency distribution	exponential	half-normal
$\alpha$	-7.7 (-8.2, -7.0)	-7.7 (-8.3, -7.0)
$\beta_1$	0.44 (0.35, 0.52)	0.44 (0.35, 0.52)
$\beta_2$	0.27 (0.16, 0.37)	0.27 (0.15, 0.39)
$\beta_3$	0.06 (0.00, 0.16)	0.06 (0.00, 0.17)
$\beta_4$	0.028 (0.023, 0.034)	0.028 (0.023, 0.034)
$\nu$	4.4 (2.1, 11.8)	3.6 (1.6, 10.4)
$\lambda$	11.3 (7.1, 29.4)	43.6 (20.6, 140.9)
$\sigma^2$	0.008 (0.003, 0.016)	0.006 (0.001, 0.014)

Table 7: Parameter results (posterior medians and 95% credible intervals) for two inefficiency distributions with  $t$  errors for the electricity data with economic restrictions

It should be noted that the non-standard likelihood function can lead to a severe deterioration in the performance of the Gibbs sampler that WinBUGS implements, which can lead to slow convergence and mixing. In this example, we used a thinning of 100 which lead to good autocorrelation properties in the chain. Table 7 presents results for two possible choice of inefficiency distribution. Comparison with Table 5 shows that the economic constraints are rarely violated for these data and the implementation of economic constraints in this example has little effect on the analysis. Only the inference on  $\beta_3$  is moderately affected.

## 4 A Panel of US Hospital data

Our second example reanalyses data on costs of US hospitals initially conducted in Koop, Osiewalski and Steel (1997), and we refer to the latter paper for further details and background of hospital cost

estimation, the data and the particular frontier used. The data correspond to  $N = 382$  nonteaching U.S. hospitals over the years 1987-1991 ( $T = 5$ ), selected so as to constitute a relatively homogeneous sample. The frontier describing cost involves five different outputs  $Y_1, \dots, Y_5$ : number of cases, number of inpatient days, number of beds, number of outpatient visits and a case mix index. We also include a measure of capital stock,  $C$ , an aggregate wage index,  $P$ , and a time trend  $t$  to capture any missing dynamics. We choose a flexible translog specification and impose linear homogeneity in prices, which allows us to normalize with respect to the price of materials. Thus, in the notation of (1) and dropping observational subscripts for ease of notation,  $y = \ln(\text{cost})$  and  $x'\beta$  becomes:

$$\begin{aligned} x'\beta = & \sum_{i=1}^5 \beta_i \ln Y_i + \beta_6 \ln P + \beta_7 (\ln P)^2 + \sum_{i=1}^5 \beta_{7+i} \ln Y_i \ln P + \beta_{13} \ln C \\ & + \sum_{i=1}^5 \beta_{13+i} \ln Y_i \ln C + \beta_{19} \ln P \ln C + \beta_{20} (\ln C)^2 + \beta_{21} t + \beta_{22} t^2 \\ & + \sum_{i=1}^5 \sum_{j=i}^5 \beta_{22+5(i-1)+j} \ln Y_i \ln Y_j. \end{aligned}$$

Throughout this section, we will use the normal sampling in (1) combined with an exponential inefficiency distribution.

#### 4.1 Including covariates in the inefficiency distribution

Koop, Osiewalski and Steel (1997) consider a method for extending the stochastic frontier to allow exponentially distributed inefficiencies to depend upon covariates. Their model assumes that each firm has a vector of binary covariates,  $w_i$  for the  $i$ -th firm. That firm's inefficiency is modelled as

$$u_i \sim \text{Exp}(\exp\{w_i \gamma\}). \quad (4)$$

where  $\gamma$  is a vector of regression coefficients. In the current example, there are 3 possible ownership categories for each firm and so  $w$  contains dummy variables to represent category membership. Actually, since WinBUGS does not rely on known forms for the conditionals, the covariates  $w_i$  can also include non-binary variables. It seems reasonable to assume *a priori* that our belief about the efficiency distribution for each category should be the same, *i.e.*

$$\exp\{\gamma_j\} \sim \text{Exp}(-\log r^*).$$

This model can be coded by defining `lambda[i]`, the inverse of mean inefficiency for the  $i$ -th firm and `data2` a matrix containing each firm's characteristics  $w$ .

```
for (i in 1:N) {
  lambda[i] <- exp(inprod(gamma[], data2[i, 1:p2]))

  u[i] ~ dexp(lambda[i])
  eff[i] <- exp(- u[i])
}
```

It is easier to define a prior distribution for  $\exp\{\gamma_j\}$  and WinBUGS allows us to define the relationship  $\text{gamma}[j] \leftarrow \log(\text{expgamma}[j])$  using a logical node. The WinBUGS code for this prior is:

```
lambda0 <- -log(rstar)
for ( j in 1:p2 ) {
  gamma[j] <- log(expgamma[j])
  expgamma[j] ~ dexp(lambda0)
}
```

Using ownership dummies (indicating non-profit, for-profit or government-run hospitals) as covariates for the inefficiency distribution, Table 9 compares the DIC values for the basic model with that of the model including these covariates. There seems to be little support for this model extension. Nevertheless, Table 8 does indicate some evidence for lower efficiencies in the for-profit sector (corresponding to  $\gamma_2$ ). The latter is in line with the results in Koop, Osiewalski and Steel (1997).

	basic model	exogeneous variables	time-varying efficiencies
$\lambda$	5.75 (5.07, 6.51)		6.00 (5.22, 6.87)
$\exp\{\gamma_1\}$		6.25 (5.38, 7.29)	
$\exp\{\gamma_2\}$		4.04 (3.12, 5.16)	
$\exp\{\gamma_3\}$		6.89 (5.11, 9.16)	
$\eta$			-0.026 (-0.047, -0.0041)
$\sigma^2$	0.0042 (0.0039, 0.0045)	0.0042 (0.0039, 0.0045)	0.0042 (0.0039, 0.0045)

Table 8: Selected parameter results (posterior medians and 95% credible intervals) for the hospital data

	$\bar{D}$	$\hat{D}$	$p_D$	DIC
basic model	-5033	-5413	379.7	-4654
exogeneous variables	-5025	-5403	378.0	-4647
time-varying	-5041	-5423	381.5	-4660

Table 9: DIC results for the hospital data with various model specifications

## 4.2 Time-varying efficiency

The assumption made in equation (2) that firm-specific technical efficiency is constant over time may not always be tenable. An alternative model that allows time-varying efficiencies uses  $u_{it}$  to represent the inefficiency of firm  $i$  at time  $t$ . A simplifying assumption proposed in Lee and Schmidt (1993) defines  $u_{it} = \beta(t)u_i$ . This specification is parsimonious but makes a strong assumption about the form of time-dependence. Several forms have been considered for  $\beta(t)$ , in particular Battese and Coelli (1992) propose

$$\beta(t) = \exp\{\eta(t - T)\}$$

where positive  $\eta$  indicates increasing efficiency over time. We choose the prior distribution of  $\eta$  to be a zero-mean normal distribution with variance 0.25 which represents our prior indifference between increasing and decreasing efficiency and, for  $T = 5$ , supports reasonable predictive distributions of efficiency at each time point. This new model can be implemented in WinBUGS by changing the mean of the log costs to

```
t <- data[k, p+2]
mu[k] <- alpha + exp(eta*(t-T)) * u[firm[k]] + inprod(beta[], data[k, 1:p])
```

and the prior is represented by the statement

```
eta ~ dnorm(0.0, 4).
```

Table 9 indicates some support for this time-varying model over the basic model. From the posterior results on  $\eta$  in Table 8, we conclude that efficiencies tend to decrease somewhat over time.

### 4.3 Model comparison of the form of the frontier

The stochastic frontier represents an approximation to the best-practice output for a set of inputs or lowest cost for producing a set of outputs. The common forms of the function have usually been log-linear but there may be uncertainty in the frontier specification. For example, a Cobb-Douglas functional form is fairly restricted. A translog is much more flexible, using a second order approximation to the unknown function, but at the expense of introducing many parameters which can lead to a poor fit. An improvement in fit may arise through careful selection of the higher order terms and interactions or of various other covariates. In the example above, we might wish to allow for the possibility of no time trend, a linear time trend or a quadratic time trend. In the Bayesian approach these questions of model choice can be answered through model comparison tools such as the DIC. We restrict attention to deciding on the type of time trend to include in the model. Table 10 shows the results of running three separate chains incorporating no time trend, a linear trend and a quadratic time trend in the basic model with a common efficiency distribution for all hospital types and constant efficiencies over time. The models with a time trend far outperform the model without time trend. The results indicate a preference for the model with a quadratic time effect, as used in the previous subsections.

	$\bar{D}$	$\hat{D}$	$p_D$	$DIC$
quadratic time trend	-5033	-5413	379.7	-4654
linear time trend	-5026	-5405	379.2	-4647
no time trend	-2581	-2914	333.4	-2248

Table 10: DIC results for the hospital data with the basic model

## 5 Conclusions

This paper demonstrates that WinBUGS provides a useful framework for the Bayesian analysis of stochastic frontier models. We provide code to implement a standard model for cross-sectional, balanced and unbalanced panel data with a time-invariant exponential inefficiency distribution. Many other, more complicated, models can be analysed using simple extensions of this code, for example to include covariates in the efficiency distribution and time-varying efficiencies. We also illustrate ways to impose economic regularity conditions and deal with model uncertainty. We emphasize that we have merely shown a few possible model specifications and extensions<sup>7</sup> and it should also be fairly straightforward to implement many other Bayesian models such as random coefficient frontiers (see Tsionas, 2002 and Huang, 2004) or models developed in the literature for dealing with multiple-output analysis or the modelling of undesirable outputs (see Fernández, Koop and Steel, 2002). We also stress that priors can be specified in line with genuine prior beliefs, as WinBUGS does not require priors to be conjugate in any sense. Modelling, both of the sampling model and the prior assumptions, can be conducted creatively and in accordance with the particular problem at hand, without worrying about having to develop and modify complicated computer code.

WinBUGS immediately leads to full posterior distributions of the model parameters and interesting functions of these parameters, such as firm-specific efficiencies and the rankings of firm-specific efficiencies. Many graphical and other summaries of the posterior distributions and the behaviour of the MCMC sampler are built-in. The availability of CODA-compatible output allows a range of convergence diagnostics to be produced very easily.

We would certainly recommend the applied user of stochastic frontier models to experiment with WinBUGS and we hope that the availability of WinBUGS code will allow these users to add Bayesian methods to their modelling and inference toolbox.

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<sup>7</sup>Although our paper covers perhaps the most popular models for typical applications.

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