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**A BIVARIATE MARKOV REGIME
SWITCHING GARCH APPROACH
TO ESTIMATE TIME VARYING
MINIMUM VARIANCE HEDGE
RATIOS**

by

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A Bivariate Markov Regime Switching GARCH Approach to Estimate Time
Varying Minimum Variance Hedge Ratios

ABSTRACT

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Keywords: bivariate GARCH, require switching, hedging

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Abstract

This paper develops a new bivariate Markov regime switching BEKK-GARCH (RS-BEKK-GARCH) model. The model is a state-dependent bivariate BEKK-GARCH model, and an extension of Gray's univariate generalized regime-switching (GRS) model to the bivariate case. To solve the path-dependency problem inherent in the bivariate regime switching BEKK-GARCH model, we propose a recombining method for the covariance term in the conditional variance-covariance matrix. The model is applied to estimate time-varying minimum variance hedge ratios for corn and nickel spot and futures prices. Out-of-sample point estimates of hedging portfolio variance show that compared to the state-independent BEKK-GARCH model, the RS-BEKK-GARCH model improves out-of-sample hedging effectiveness for both corn and nickel data. We perform White's (2000) data-snooping reality check to test for predictive superiority of RS-BEKK-GARCH over the benchmark model, and find that the difference in variance reduction between BEKK-GARCH and RS-BEKK-GARCH is not statistically significant for either data set at conventional confidence levels.

I. Introduction

If the true hedge ratio that minimizes the variance of a hedging portfolio is constant, then the slope coefficient of an Ordinary Least Squares regression of spot returns on

futures returns is an appropriate estimate of the optimal hedge ratio (Ederington, 1979; Figlewski, 1984). However, if the true joint distribution of spot and futures returns and hence the hedge ratio is changing through time, the (constant) OLS slope coefficient may be inferior to more flexible models (Park and Switzer, 1995).

To estimate time-varying optimal hedge ratios, a considerable amount of research has applied the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models proposed by Engle (1982), Bollerslev (1986, 1990), and others. Various GARCH models have been applied to investigate foreign exchange rate futures (Kroner and Sultan, 1993; Gagnon, Lypny, and McCurdy, 1998), interest rate futures (Gagnon and Lypny, 1995; Cecchetti, Cumby, and Figlewski, 1988), stock index futures (Park and Switzer, 1995; Tong, 1996; Brooks, Henry, and Persaud, 2002), and commodity futures (Baillie and Myers, 1991; Myers, 1991; Bera, Garcia, and Roh, 1997; Byström, 2003).

This paper contributes to this line of research by proposing a new model that extends the BEKK-GARCH framework (henceforth BEKK) of Engle and Kroner (1995) to allow regime shifts into the by allowing model parameters to be affected by the state of the market. The regime changes are governed by an unobserved state variable that follows a first-order, two-state Markov process. As a consequence, the hedge ratio estimated from our Regime-Switching BEKK-GARCH model (RS-BEKK-GARCH; henceforth RS-BEKK for brevity) is both time varying and state-dependent.¹

RS-BEKK can also be viewed as an extension of Gray's (1996) univariate generalized regime-switching (GRS) model to the bivariate case. Gray's GRS model is a general regime-switching model that allows for GARCH innovations. When the GARCH

process is subject to regime switching, however, the basic form of the model is intractable due to a well-known path-dependency problem (Cai, 1994; Hamilton and Susmel, 1994; Gray, 1995, 1996). Path-dependency occurs when the conditional variance at time t depends on the entire sequence of regimes up to time t due to the recursive nature of the GARCH process and state dependent GARCH coefficients. Gray solves the path-dependency problem by introducing a recombining method that collapses the conditional variances in each regime into a single variance at each point in time. By doing so, the model becomes path-independent and is tractable even with large sample size. However, estimation of minimum variance hedge ratios requires estimation of variances and covariance of spot and futures returns. Gray's univariate GRS model cannot do this. Our bivariate generalization of Gray's model can, but to do so, the path dependency problem must be resolved for the conditional variances (as in Gray's univariate model), as well as the conditional covariance of the spot and futures returns. For the variance terms, we apply Gray's recombining methods for both spot and futures returns. We then extend Gray's recombining method for the conditional covariance of spot and futures returns to completely solve the path-dependency problem encountered in RS-BEKK.

RS-BEKK is different from the switching BEKK model proposed by Gannon and Au-Yeung (2004). Gannon and Au-Yeung allow the bivariate GARCH variance-covariance structure to be subject to a finite number of shifts implemented by adding event dummy variables in the GARCH process. Often however, the timing of regime changes are unknown to researchers *a priori*. In contrast to their switching BEKK model, RS-BEKK estimates the switching points instead of imposing them. In our model, the regime

shifts are governed by a latent state variable that follows a first-order, two-state Markov process, the parameters of which is estimated via maximum likelihood along with other unknown system parameters.

We apply RS-BEKK to two futures contracts, corn and nickel, traded on the Chicago Board of Trade and the London Metal Exchange, respectively. We compare its performance to the state-independent BEKK model and Ordinary Least Square (OLS) based on variance reduction of the hedged portfolio of each model. Based on out-of-sample point estimates of hedging performance, RS-BEKK is superior to BEKK for both corn and nickel. In-sample, RS-BEKK is superior to BEKK for nickel but is inferior to BEKK for corn. To test statistical significance of these differences in hedging performance, we apply White's data-snooping reality check (White, 2000). The null hypothesis that the performance of the best dynamic hedging model (RS-BEKK) has no predictive superiority over the BEKK model is not rejected for both corn and nickel futures contracts. Thus, point estimates suggest that RS-BEKK perform well relative to BEKK, but not significantly so.

In the next section, the bivariate BEKK-GARCH model is summarized, and section III summarizes Gray's GRS model. In section IV, we present RS-BEKK. Hedging performance criteria, and White's data snooping reality check test are discussed in section V, and data descriptions and empirical results are reported and discussed in section VI. Section VII concludes.

II. BEKK GARCH

Bivariate GARCH models are widely used in studying the time-varying minimum variance hedge ratio. The bivariate BEKK model used in this study is specified below (Bera, Garcia, and Roh, 1997). Let $r_{c,t}$ and $r_{f,t}$ be the returns on the spot and futures, respectively. BEKK- GARCH is specified as

$$r_{c,t} = \mu_c + e_{c,t} \quad (1)$$

$$r_{f,t} = \mu_f + e_{f,t}, \quad (2)$$

$$\mathbf{e}_t | \Psi_{t-1} = \begin{bmatrix} e_{c,t} \\ e_{f,t} \end{bmatrix} | \Psi_{t-1} \sim BN(0, \mathbf{H}_t), \quad (3)$$

where subscripts c and f denote cash and futures prices, $e_{c,t}$ and $e_{f,t}$ are disturbances, Ψ_{t-1} refers to the information available at time $t-1$, BN denotes the bivariate normal density function, and \mathbf{H}_t is a time-varying 2×2 positive definite conditional covariance matrix, specified as

$$\begin{aligned} \mathbf{H}_t &= \begin{bmatrix} h_{c,t}^2 & h_{cf,t} \\ h_{cf,t} & h_{f,t}^2 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{cc} & 0 \\ \gamma_{fc} & \gamma_{ff} \end{bmatrix} \begin{bmatrix} \gamma_{cc} & 0 \\ \gamma_{fc} & \gamma_{ff} \end{bmatrix}' + \begin{bmatrix} \alpha_{cc} & \alpha_{cf} \\ \alpha_{fc} & \alpha_{ff} \end{bmatrix}' \begin{bmatrix} e_{c,t-1}^2 & e_{c,t-1}e_{f,t-1} \\ e_{c,t-1}e_{f,t-1} & e_{f,t-1}^2 \end{bmatrix} \begin{bmatrix} \alpha_{cc} & \alpha_{cf} \\ \alpha_{fc} & \alpha_{ff} \end{bmatrix} \\ &\quad + \begin{bmatrix} \beta_{cc} & \beta_{cf} \\ \beta_{fc} & \beta_{ff} \end{bmatrix}' \begin{bmatrix} h_{c,t-1}^2 & h_{cf,t-1} \\ h_{cf,t-1} & h_{f,t-1}^2 \end{bmatrix} \begin{bmatrix} \beta_{cc} & \beta_{cf} \\ \beta_{fc} & \beta_{ff} \end{bmatrix}, \quad (4) \end{aligned}$$

where $h_{cf,t}$ is a conditional covariance of spot and futures returns, $h_{c,t}^2$ and $h_{f,t}^2$ are conditional variances of spot and futures returns, respectively. The unknown parameters

are $\boldsymbol{\theta} = \{\mu_c, \mu_f, \gamma_{cc}, \gamma_{fc}, \gamma_{ff}, \alpha_{cc}, \alpha_{cf}, \alpha_{fc}, \alpha_{ff}, \beta_{cc}, \beta_{cf}, \beta_{fc}, \beta_{ff}\}$, which can be estimated by maximizing the following log-likelihood function with respect to $\boldsymbol{\theta}$:

$$L(\boldsymbol{\theta}) = -T \log(2\pi) - \frac{1}{2} \sum_{t=1}^T \log |\mathbf{H}_t(\boldsymbol{\theta})| - \frac{1}{2} \sum_{t=1}^T \mathbf{e}_t(\boldsymbol{\theta})' \mathbf{H}_t^{-1}(\boldsymbol{\theta}) \mathbf{e}_t(\boldsymbol{\theta}), \quad (5)$$

where T is the total number of observations. The estimated time-varying minimum variance hedge ratio $\hat{\beta}_t^*$ can be expressed with the variances and covariance estimates from (4) as

$$\hat{\beta}_t^* = \hat{h}_{cf,t} / \hat{h}_{f,t}^2. \quad (6)$$

III. Generalized Regime-Switching (GRS) Model

The GARCH family of models is a popular approach for modeling the time-varying conditional volatility, but the structural forms of the conditional means and variances of GARCH models are held fixed throughout the entire sample period. To condition the model coefficients on the state of the market, Hamilton and Susmel (1994) and Cai (1994) propose the regime-switching model with ARCH innovations in which the conditional variance process is allowed to switch among different regimes according to a latent state variable that follows a Markov process.

Gray (1996) introduced the generalized regime-switching (GRS) model, which allows for GARCH innovations. The GRS model is described as follows. Let the r_t be the return at time t , modeled as a constant plus a disturbance term such that

$$r_t = \mu_{s_t} + e_{t,s_t}, \quad (7)$$

$$e_{t,s_t} | \Psi_{t-1} = h_{t,s_t} z_t, \quad (8)$$

where $s_t = \{1, 2\}$ is an unobserved state variable at time t , which follows a first-order, two-state Markov process, e_{t,s_t} is a state-dependent residual term, z_t is a standard normal random variable, and h_{t,s_t} is a state-dependent, conditional standard deviation of r_t . The conditional volatility is assumed to follow a GARCH(1,1) process

$$h_{t,s_t}^2 = \gamma_{s_t} + \alpha_{s_t} e_{t-1}^2 + \beta_{s_t} h_{t-1}^2, \quad (9)$$

where γ_{s_t} , α_{s_t} , and β_{s_t} are state dependent coefficients.

When the GARCH process is subject to regime switching, however, the recursive nature of the GARCH process makes the basic form of the model intractable due to the dependence of the conditional variance on the entire past history of the data. This is the well-known path-dependency problem in the regime switching literature (Cai, 1994; Hamilton and Susmel, 1994; Gray, 1995, 1996). Appendix A includes a graphical expression of the evolution of conditional variances in a path-dependent GARCH model, and shows how the path-dependent model can be transformed into a path-independent model. Gray solves the path-dependency problem by introducing a recombining method that collapses the conditional variances in each regime by taking the conditional expectation of h_t^2 based on the regime probabilities, such that the path-independent variances are defined as

$$\begin{aligned} h_t^2 &= E[r_t^2 | \Psi_{t-1}] - E[r_t | \Psi_{t-1}]^2 \\ &= p_{1t} (\mu_1^2 + h_{t,1}^2) + (1 - p_{1t}) (\mu_2^2 + h_{t,2}^2) - [p_{1t} \mu_1 + (1 - p_{1t}) \mu_2]^2. \end{aligned} \quad (10)$$

As a consequence, the conditional variance depends only on the current regime, not on the entire past history of the process and the model is free from the path-dependency problem.

Similarly, the recombining method for the residual is given by

$$\begin{aligned} e_t &= r_t - E[r_t | \Psi_{t-1}] \\ &= r_t - [p_{1t}\mu_1 + (1 - p_{1t})\mu_2] \end{aligned} \quad (11)$$

where $p_{1t} = \Pr(s_t = 1 | \Psi_{t-1})$ is the regime probability of being in state 1 given all information up to time $t-1$. After recombining, the path-independent conditional variances and residuals can be used as the lagged conditional variances and residuals in constructing next period's conditional variance.

To calculate regime probabilities, Gray (1995, 1996) derived a nonlinear recursive expression of the regime probability as a function of transition probabilities and conditional distributions. This recursive expression simplifies the construction of the likelihood function and permits easy estimation of relatively complicate models:

$$\begin{aligned} p_{1t} &= \Pr(s_t = 1 | \Psi_{t-1}) \\ &= P \left[\frac{g_{1t-1} p_{1t-1}}{g_{1t-1} p_{1t-1} + g_{2t-1} (1 - p_{1t-1})} \right] + (1 - Q) \left[\frac{g_{2t-1} (1 - p_{1t-1})}{g_{1t-1} p_{1t-1} + g_{2t-1} (1 - p_{1t-1})} \right] \end{aligned} \quad (12)$$

where

$$P = \Pr[s_t = 1 | s_{t-1} = 1], \quad (13)$$

$$Q = \Pr[s_t = 2 | s_{t-1} = 2], \quad (14)$$

$$g_{it} = f(r_t | s_t = i, \Psi_{t-1}) = \frac{1}{\sqrt{2\pi} h_{t,i}} \exp \left\{ -\frac{(r_t - \mu_i)^2}{2 h_{t,i}^2} \right\}, \quad i = 1, 2. \quad (15)$$

P and Q are transition probabilities, which are the probabilities that the regime 1 and 2 at time $t-1$ followed by regime 1 and 2 at time t , respectively, and g_{it} is the conditional probability density function of the return given that state i occurs at time t and given all information available up to time $t-1$. The parameters $\theta = \{P, Q, \mu_{s_t}, \gamma_{s_t}, \alpha_{s_t}, \beta_{s_t}\}$, for $s_t = 1, 2$ can be estimated by maximizing the following log-likelihood function with respect to θ :

$$L(\theta) = \sum_{t=1}^T \log \left\{ p_{1t} \frac{1}{\sqrt{2\pi h_{t,1}}} \exp \left[\frac{-(r_t - \mu_1)^2}{2h_{t,1}^2} \right] + (1 - p_{1t}) \frac{1}{\sqrt{2\pi h_{t,2}}} \exp \left[\frac{-(r_t - \mu_2)^2}{2h_{t,2}^2} \right] \right\}, \quad (16)$$

IV. Bivariate Regime Switching GARCH model

RS-BEKK nests within it both Gray's univariate GRS model and the state-independent BEKK model. The state-dependent cash and futures returns are specified as

$$r_{c,t} = \mu_{c,s_t} + e_{c,t,s_t} \quad (17)$$

$$r_{f,t} = \mu_{f,s_t} + e_{f,t,s_t}, \quad (18)$$

and their conditional covariance are specified as

$$\mathbf{e}_{t,s_t} | \boldsymbol{\psi}_{t-1} = \begin{bmatrix} e_{c,t,s_t} \\ e_{f,t,s_t} \end{bmatrix} | \boldsymbol{\psi}_{t-1} \sim BN(\mathbf{0}, \mathbf{H}_{t,s_t}), \quad (19)$$

where $s_t = \{1, 2\}$ is the state variable indicating the market regime at time t , which follows a first-order, two-state Markov process. State transition probabilities are assumed to follow a logistic distribution such that

$$\Pr(s_t = 1 | s_{t-1} = 1) = P = \frac{\exp(p_0)}{1 + \exp(p_0)}, \quad (20)$$

$$\Pr (s_t = 2 \mid s_{t-1} = 2) = Q = \frac{\exp (q_0)}{1 + \exp (q_0)}, \quad (21)$$

where p_0 and q_0 are unconstrained constants to be estimated along with the other unknown system parameters via maximum likelihood, e_{c,t,s_t} and e_{f,t,s_t} are disturbances given state s_t at time t , and BN denotes the bivariate normal. \mathbf{H}_{t,s_t} is a state-dependent time-varying 2×2 positive definite conditional covariance matrix specified as

$$\begin{aligned} \mathbf{H}_{t,s_t} &= \begin{bmatrix} h_{c,t,s_t}^2 & h_{cf,t,s_t} \\ h_{cf,t,s_t} & h_{f,t,s_t}^2 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{cc,s_t} & 0 \\ \gamma_{fc,s_t} & \gamma_{ff,s_t} \end{bmatrix} \begin{bmatrix} \gamma_{cc,s_t} & 0 \\ \gamma_{fc,s_t} & \gamma_{ff,s_t} \end{bmatrix}' + \begin{bmatrix} \alpha_{cc,s_t} & \alpha_{cf,s_t} \\ \alpha_{fc,s_t} & \alpha_{ff,s_t} \end{bmatrix} \begin{bmatrix} e_{c,t-1}^2 & e_{c,t-1}e_{f,t-1} \\ e_{c,t-1}e_{f,t-1} & e_{f,t-1}^2 \end{bmatrix} \begin{bmatrix} \alpha_{cc,s_t} & \alpha_{cf,s_t} \\ \alpha_{fc,s_t} & \alpha_{ff,s_t} \end{bmatrix}' \\ &\quad + \begin{bmatrix} \beta_{cc,s_t} & \beta_{cf,s_t} \\ \beta_{fc,s_t} & \beta_{ff,s_t} \end{bmatrix} \begin{bmatrix} h_{c,t-1}^2 & h_{cf,t-1} \\ h_{cf,t-1} & h_{f,t-1}^2 \end{bmatrix} \begin{bmatrix} \beta_{cc,s_t} & \beta_{cf,s_t} \\ \beta_{fc,s_t} & \beta_{ff,s_t} \end{bmatrix}' \\ &= \mathbf{\Gamma}_{s_t} \mathbf{\Gamma}_{s_t}' + \mathbf{A}_{s_t} \mathbf{E}_{t-1} \mathbf{A}_{s_t}' + \mathbf{B}_{s_t} \mathbf{H}_{t-1} \mathbf{B}_{s_t}' \end{aligned} \quad (22)$$

where h_{cf,t,s_t} is a conditional covariance at time t given s_t , and h_{c,t,s_t}^2 and h_{f,t,s_t}^2 are conditional variances at time t given s_t . The matrices $\mathbf{\Gamma}_{s_t}$, \mathbf{A}_{s_t} , and \mathbf{B}_{s_t} and \mathbf{E}_{t-1} are compact representations of the γ 's, α 's, β 's and e 's, respectively.

As in the univariate regime switching GARCH model, the proposed bivariate GARCH model is also subject to the path-dependency problem and is intractable in its basic form. Furthermore, RS-BEKK is a bivariate model, so we must collapse not only variances and residuals, but also the covariances of spot and futures returns. Below, we first briefly revisit Gray's approach for variances and residuals, then we describe an

analogous approach for the covariances. A graphical illustration for the recombining method for RS-BEKK model is shown in appendix A.

Gray's recombining method for collapsing the conditional variances and residuals for each regime into a single value at each point in time as applied to both cash and futures prices can be characterized as

$$h_{i,t}^2 = p_{1t}(\mu_{i,1}^2 + h_{i,t,1}^2) + (1 - p_{1t})(\mu_{i,2}^2 + h_{i,t,2}^2) - [p_{1t}\mu_{i,1} + (1 - p_{1t})\mu_{i,2}]^2, \quad (23)$$

$$e_{i,t} = r_{i,t} - E[r_{i,t} | \Psi_{t-1}] = r_{i,t} - [p_{1t}\mu_{i,1} + (1 - p_{1t})\mu_{i,2}], \quad i = \{c, f\}. \quad (24)$$

The regime probability of being in state 1 at time t is

$$\begin{aligned} p_{1t} &= \text{Pr}(s_t = 1 | \Psi_{t-1}) \\ &= P \left[\frac{f_{1t-1} p_{1t-1}}{f_{1t-1} p_{1t-1} + f_{2t-1} (1 - p_{1t-1})} \right] + (1 - Q) \left[\frac{f_{2t-1} (1 - p_{1t-1})}{f_{1t-1} p_{1t-1} + f_{2t-1} (1 - p_{1t-1})} \right] \end{aligned} \quad (25)$$

where

$$f_{it} = f(\mathbf{R}_t | s_t = i, \Psi_{t-1}) = (2\pi)^{-1} |\mathbf{H}_{t,i}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mathbf{e}'_{t,i} \mathbf{H}_{t,i}^{-1} \mathbf{e}_{t,i} \right\}, \quad \text{for } i = \{1, 2\}, \quad (26)$$

and $\mathbf{R}_t = [r_{c,t} \quad r_{f,t}]'$ is a vector of spot and futures returns at time t . \mathbf{H} 's and \mathbf{e} 's are defined in equation (19) and (22), respectively. The recursive expression of the regime probability shown in equation (25) is derived in appendix B. The proof is the same as that proposed by Gray (1995, 1996), but with the conditional univariate normal distribution replaced by a conditional bivariate normal distribution.

The steady-state probabilities of s_t used as the initial start value for the recursive expression of the regime probability is

$$\Pr (s_1 = 1 | \psi_0) = \frac{1 - Q}{2 - P - Q}, \quad (27)$$

where P and Q are transition probabilities defined in equation (20) and (21).

To fully solve the path-dependency problem, we also need to collapse the conditional covariance of spot and futures returns. We propose the following recombining method, which extends Gray's approach to covariances. Define

$$\begin{aligned} h_{cf,t} &= Cov(r_{c,t}, r_{f,t} | \Psi_{t-1}) \\ &= E[r_{c,t}r_{f,t} | \Psi_{t-1}] - E[r_{c,t} | \Psi_{t-1}]E[r_{f,t} | \Psi_{t-1}]. \end{aligned} \quad (28)$$

The conditional expectations $E[r_{c,t}r_{f,t} | \Psi_{t-1}]$, $E[r_{c,t} | \Psi_{t-1}]$, $E[r_{f,t} | \Psi_{t-1}]$ are defined in terms of estimable parameters as follows:

$$E[r_{c,t} | \Psi_{t-1}] = p_{1t}\mu_{c,1} + (1 - p_{1t})\mu_{c,2}, \quad (29)$$

$$E[r_{f,t} | \Psi_{t-1}] = p_{1t}\mu_{f,1} + (1 - p_{1t})\mu_{f,2}, \quad (30)$$

$$\begin{aligned} E[r_{c,t}r_{f,t} | \Psi_{t-1}] &= E[(\mu_{c,s_t} + e_{c,t,s_t})(\mu_{f,s_t} + e_{f,t,s_t}) | \Psi_{t-1}] \\ &= p_{1t}(\mu_{c,1}\mu_{f,1} + h_{cf,t,1}) + (1 - p_{1t})(\mu_{c,2}\mu_{f,2} + h_{cf,t,2}). \end{aligned} \quad (31)$$

With this definition, the conditional covariance depends only on the current regime, not on the entire past history of the process. The model is then state-independent and tractable even with large samples.

Given the structure of RS-BEKK and the recombining approach described above, the unknown parameters for the full estimation problem are $\{p_0, q_0, \mu_{c,s_t}, \mu_{f,s_t}, \gamma_{cc,s_t}, \gamma_{fc,s_t}, \gamma_{ff,s_t}, \alpha_{cc,s_t}, \alpha_{cf,s_t}, \alpha_{fc,s_t}, \alpha_{ff,s_t}, \beta_{cc,s_t}, \beta_{cf,s_t}, \beta_{fc,s_t}, \beta_{ff,s_t}\}$ for $s_t = \{1, 2\}$. These parameters can be estimated by recursively solving the likelihood function

$$LL = \sum_{t=1}^T \log[p_{1t}f_{1t} + (1 - p_{1t})f_{2t}] \quad (32)$$

where f_{it} for $i = \{1,2\}$ are defined in equation (26). Time-varying minimum variance hedge ratios are calculated with variances and covariance estimates from (23) and (28) as

$$\hat{\beta}_t^* = h_{cf,t} / h_{f,t}^2. \quad (33)$$

V. Measuring Hedging Performance

The hedging performance is typically evaluated based on the variance reduction of the hedged portfolio relative to the unhedged position. The variance of the estimated optimal hedged portfolio can be expressed as

$$Var\left(r_{c,t} - \hat{\beta}_t^* r_{f,t}\right), \quad (34)$$

where $\hat{\beta}_t^*$'s are the estimated optimal hedge ratios derived from OLS, BEKK, and RS-BEKK models. The percentage variance reduction is calculated based on equation (34) as $100 \cdot \frac{Var[\text{unhedged}] - Var[\text{hedged}]}{Var[\text{unhedged}]}$.

In addition to providing a measure of risk reduction, we also test the statistical significance of the variance reduction by applying the bootstrap version of White's reality check for data snooping (White, 2000, Sullivan, Timmermann, and White 1999). Data snooping bias might occur when a given dataset is reused by one or more researchers for model selection. White's reality check is used for testing the null hypothesis that the best model encountered in a specification search has no predictive superiority over a given

benchmark model. The innovation of White's method is that it uses information provided by existing alternative models with intermediate performance to statistically assess the performance of the best-performing model. White's reality check is based on the following $l \times 1$ performance statistic:²

$$\bar{f} = n^{-1} \sum_{t=R}^T \hat{f}_{t+1} \quad (35)$$

where l is the number of alternative models, n is the number of prediction periods indexed from R to T so that $n = T - R + 1$, \hat{f}_{t+1} is the observed performance measure for period $t + 1$ and is defined as:

$$\hat{f}_{t+1} = -\left(r_{c,t} - \hat{\beta}_{k,t} r_{f,t}\right)^2 + \left(r_{c,t} - \hat{\beta}_{BM} r_{f,t}\right)^2 \quad (36)$$

where $\hat{\beta}_{BM}$ is the estimate of β from the benchmark, and $\hat{\beta}_{k,t}$, $\forall k = 1, \dots, l$ is the one-step-ahead prediction of β from alternative models at time t . The null hypothesis that the performance of the best dynamic hedging model is no better than the benchmark:

$$H_0 : \max_{k=1, \dots, l} \left[E \left(f_k^* \right) \right] \leq 0, \quad (37)$$

where f_k^* is the performance value for each model applied to the data. Following White (2000), we base the test on the stationary bootstrap resampling method of Politis and Romano (1994) applicable to time series data, with which pseudo-time series are generated by resampling blocks of random size where the length of each block has a geometric distribution. This resampling procedure is repeated to generate an

approximation to the sampling distribution of a statistic of interest, which in our case is the performance measurement, namely the \bar{f} in equation (35).

VI. Data Description and Empirical Results

For performance comparisons, optimal hedging portfolios are generated with RS-BEKK, BEKK, and OLS using two futures contracts, corn and nickel, traded in the Chicago Board of Trade and the London Metal exchange. The spot and futures data are Wednesday's closing price for the nearby contract. Tuesday's closing price is used when a holiday occurs on Wednesday. The full sample period is from 01/02/1991 to 12/29/2004. The data for the period 01/02/1991 to 12/31/2003 are used for estimation and in-sample forecasts, and the data for the period 01/07/2004 to 12/29/2004 are used for out-of-sample forecasts. The spot and futures returns are calculated as the first difference in the logarithm of price multiplied by 100. Summary statistics for spot and futures prices of corn and nickel are shown in Table I.

The parameter estimates for the alternative models are presented in table II. The simulations were performed using GAUSS version 6.0 and the parameters are those of BEKK and RS-BEKK estimated by maximizing the log-likelihood functions in equation (5) and (32) using the GAUSS numerical constrained optimization (CO) procedure.

Table III provides point estimates for in- and out-of-sample hedging effectiveness of the alternative models for corn and nickel. RS-BEKK has 78.89% and 99.21% variance reduction for corn and nickel, respectively. These are better than that of BEKK, which has 76.98% variance reduction for corn and 99.20% variance reduction for nickel.

Based on in-sample data, RS-BEKK provides variance reduction of 98.74% for nickel, which is slightly better than BEKK, which provides 98.73% variance reduction. RS-BEKK provides 62.20% variance reduction for corn, which is inferior to BEKK, with 63.00% variance reduction. Both BEKK and RS-BEKK are superior to OLS in-sample. Out of sample, RS-BEKK is superior to OLS, which is superior to BEKK.

To test the statistical significance of the performance improvements of these dynamic hedging models, we perform White's reality check as described in section V.³ When BEKK is treated as the benchmark, we find that the null hypothesis of no improvement of RS-BEKK over the benchmark cannot be rejected for both corn and nickel data, based on reality check p-values of 0.315 and 0.257 for corn and nickel, respectively.

Figure 1 compares the hedge ratios of RS-BEKK, BEKK, and OLS for corn. The OLS hedge ratio is a constant and the hedge ratios estimated from the GARCH models are all time varying. Figure 2 shows the RS-BEKK estimates of the probability of being in state 1. The similar results for nickel data are shown in Figure 3 and Figure 4.

VII. Conclusions

In this article we propose a new bivariate Markov regime switching GARCH model to estimate the time-varying minimum variance hedge ratio. Our method, RS-BEKK, generalizes the BEKK-GARCH model to allow for regime shifts, and generalizes Gray's univariate GARCH model to the bivariate case by proposing a recombining method for the covariance term in the conditional variance-covariance matrix to solve the path-dependency problem.

In this study, we investigate whether allowing the BEKK model to be subject to regime shifts improves futures hedging performance. We find that for the corn and nickel futures contracts used in this paper, allowing the variance-covariance structure to be state dependent improves point estimates out-of-sample hedging effectiveness, but not statistically significantly so, based on White's Reality Check (White 2000).

This is the first paper that incorporates Markov regime shifts into the multivariate GARCH time-varying variance-covariance process. The proposed Markov regime switching BEKK GARCH model provides a very general framework in studying time-varying volatility, and the comparisons with two other commonly applied models are promising, though inconclusive based on the data used in this study.

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Endnotes

¹ Alizadeh and Nomikos also (2004) propose a Markov regime switching approach (Hamilton, 1989) for hedging stock indices. Instead of estimating the hedge ratio by estimating the conditional second moments as all GARCH methods do (including RS-BEKK-GARCH), they treat the hedge ratio as a time-varying regression coefficient, which conditions on the state of market volatility with transition probabilities a function of lagged time-varying basis and estimate the coefficient directly. The rationale behind their model is that the dynamic relationship between spot and futures returns, and hence the hedge ratio, can be characterized by regime shifts (Sarno and Valente, 2000).

² For ease of comparison and reference, we follow the notation of White (2000) as closely as possible in this section. The values referred to be the symbols f and R in this section are unrelated to those in previous sections of this paper.

³ To apply the stationary bootstrap method of Politis and Romano (1994), we set the smoothing parameter q to 0.5 and we resample 1000 times for each application. Testing for statistical significance of point estimates of hedging performance differences is relatively uncommon. Bystrom (2003) tests the statistical significance of the hedged portfolio variance by using conventional bootstrap method and finds that no hedge method differs in a statistical way from the unhedged spot position and no hedge method significantly differs from any other hedge method. By performing White's reality check, however, we can test the statistical significance of the hedging performance by incorporating the potential effect of data snooping bias.

Tables

Table I

Summary Statistics for Spot and Futures Prices of Corn and Nickel Futures Contracts

Corn								
	In Sample				Out of Sample			
	Log Level		% Return		Log Level		% Return	
	Spot	Futures	Spot	Futures	Spot	Futures	Spot	Futures
Mean	0.8525	0.8921	0.0135	0.0088	0.8463	0.9137	-0.6820	-0.4204
SD	0.2036	0.1806	3.2537	3.2101	0.2146	0.1768	4.0253	3.6074
Skewness	1.2338	1.5327	0.0127	-1.0318	-0.1186	0.0337	-0.3766	-0.1515
Kurtosis	2.6244	3.4171	1.7424	14.3795	-1.6514	-1.6317	-0.3562	-0.5255
Nickel								
	In Sample				Out of Sample			
	Log Level		% Return		Log Level		% Return	
	Spot	Futures	Spot	Futures	Spot	Futures	Spot	Futures
Mean	8.8274	8.8327	0.1022	0.1017	9.5269	9.5198	-0.0747	-0.0645
SD	0.2363	0.2318	3.8940	3.7825	0.0961	0.0942	6.0697	5.9939
Skewness	-0.1653	-0.1623	0.2076	0.1603	-0.8553	-0.7951	-0.1180	-0.2241
Kurtosis	0.3935	0.3894	3.9085	4.2142	0.5913	0.4464	0.3103	0.2723

Table II
Estimates of Unknown Parameters of Alternative Models for Corn and Nickel Futures Contracts.
Sample Period: January 2, 1991 to December 31, 2003

	Corn		Nickel	
	BEKK	RS-BEKK	BEKK	RS-BEKK
P_0		0.7277 (0.5186)		-0.0672 (0.0789)
q_0		0.1400 (2.3106)		0.8661 (0.3993)
μ_{c1}	0.1116 (0.1160) ^a	-0.0737 (1.1304)	-0.0268 (0.0327)	-0.1676 (0.1049)
μ_{c2}		0.3596 (1.9802)		-0.0094 (0.0112)
μ_{f1}	0.1746 (0.1196)	-0.3022 (1.0892)	-0.0261 (0.0330)	-0.1505 (0.1052)
μ_{f2}		0.6837 (1.8631)		-0.0203 (0.0139)
γ_{cc1}	0.8314 (0.1803)	1.6525 (0.1951)	0.4483 (0.1320)	1.4599 (0.6556)
γ_{cc2}		0.1074 (0.7434)		0.0002 (0.0414)
γ_{fc1}	1.2096 (0.1368)	1.7094 (0.2592)	0.4809 (0.1308)	1.4022 (0.7056)
γ_{fc2}		0.1190 (2.5033)		0.0002 (0.0043)
γ_{ff1}	0.0050 (0.0400)	-0.0010 (0.0938)	0.0000 (0.0082)	0.0004 (0.0244)
γ_{ff2}		-0.0009 (0.5477)		0.0000 (0.0354)
α_{cc1}	-0.2623 (0.0652)	0.4095 (0.7155)	-0.6180 (0.1723)	0.2042 (0.2730)
α_{cc2}		-0.6481 (0.3534)		-0.8650 (0.2679)
α_{cf1}	0.2078 (0.0688)	0.2896 (0.9308)	-0.1865 (0.1751)	0.2546 (0.2934)
α_{cf2}		-0.6157 (0.7344)		-0.3420 (0.2516)
α_{fc1}	0.4378 (0.0558)	0.2896 (0.6832)	0.7534 (0.1731)	0.1939 (0.0072)
α_{fc2}		0.6165 (0.6238)		0.8209 (0.3043)
α_{ff1}	0.1936 (0.0674)	-0.0278 (0.8966)	0.3283 (0.1755)	0.1480 (0.1155)
α_{ff2}		0.3378 (1.1906)		0.3014 (0.2929)
β_{cc1}	0.8275 (0.0329)	-0.6384 (0.5768)	1.0523 (0.0660)	-1.7066 (0.7160)
β_{cc2}		1.1243 (1.0253)		-0.0351 (0.0710)
β_{cf1}	-0.0493 (0.0357)	-0.2239 (0.6950)	0.1645 (0.0698)	1.9025 (0.6334)
β_{cf2}		-0.2970 (0.7047)		0.1419 (0.0652)
β_{fc1}	0.1302 (0.0486)	0.7683 (0.4951)	-0.0699 (0.0718)	1.9025 (1.1850)
β_{fc2}		0.1375 (0.7990)		0.1419 (0.0633)
β_{ff1}	0.8834 (0.0421)	0.4295 (0.2540)	0.8140 (0.0759)	1.5049 (1.0712)
β_{ff2}		1.4814 (0.1865)		1.2110 (0.0730)
Log-L ^b	-3027.8804	-2887.9595	-1975.3332	-1927.3166

- a. Figures in parentheses are standard errors.
b. Log-L stands for log likelihood

Table III
In- and Out-of-Sample Hedging Effectiveness of Alternative Models for Corn and Nickel Futures Contracts.^a

	Corn			
	Variance ^b		Variance Reduction (%) ^c	
	In Sample	Out-of-Sample	In Sample	Out-of-Sample
Unhedged	10.5988	16.0694		
OLS	4.1817	3.6199	60.5450%	77.4732%
BEKK	3.9212	3.6988	63.0031%	76.9822%
RS-BEKK	4.0065	3.3924	62.1981%	78.8891%

	Nickel			
	Variance		Variance Reduction (%)	
	In Sample	Out-of-Sample	In Sample	Out-of-Sample
Unhedged	15.1571	37.2118		
OLS	0.1941	0.2952	98.7194%	99.2068%
BEKK	0.1932	0.2989	98.7256%	99.1968%
RS-BEKK	0.1914	0.2945	98.7369%	99.2087%

- a. The in-sample data period is from January 2, 1991 to December 31, 2003 and the out-of-sample data period is from January 7, 2004 to December 29, 2004.
- b. Variance stands for the variance of the hedged portfolio calculated based on equation (34)
- c. Percentage variance reductions are calculated as the differences of variance of unhedged position and estimated variance of alternative models over variance of unhedged position multiplied by 100.

Figures

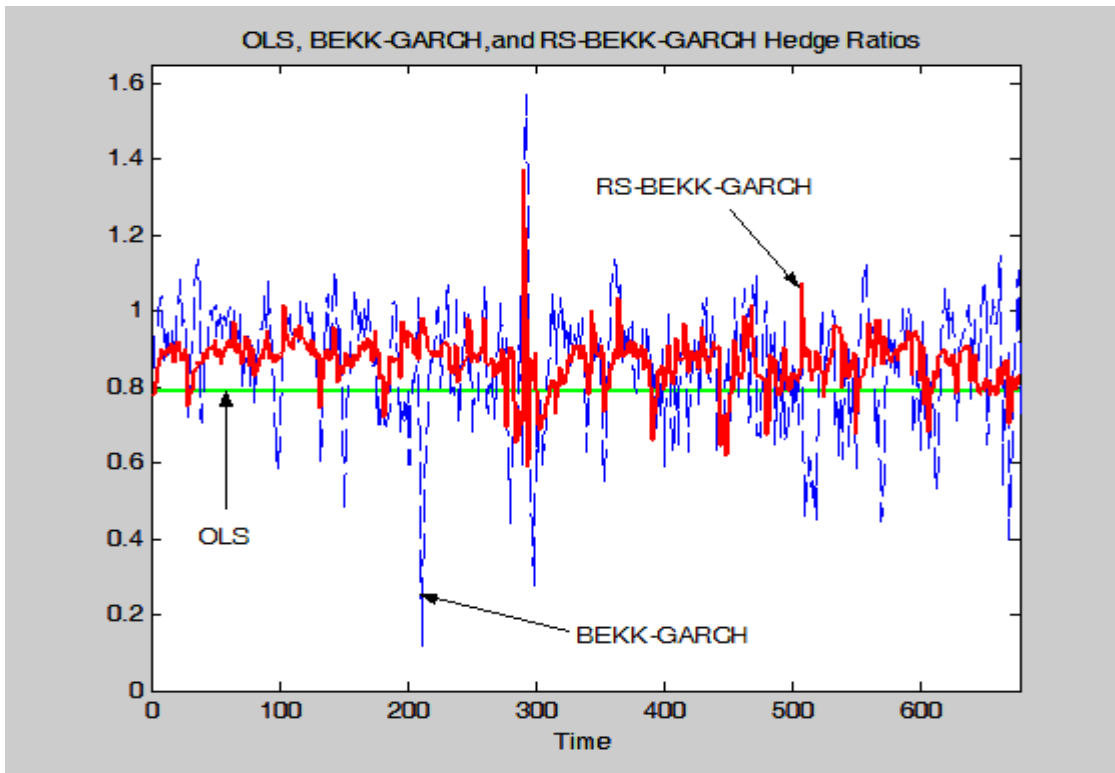


Figure 1
RS-BEKK, BEKK and OLS hedge ratios for corn

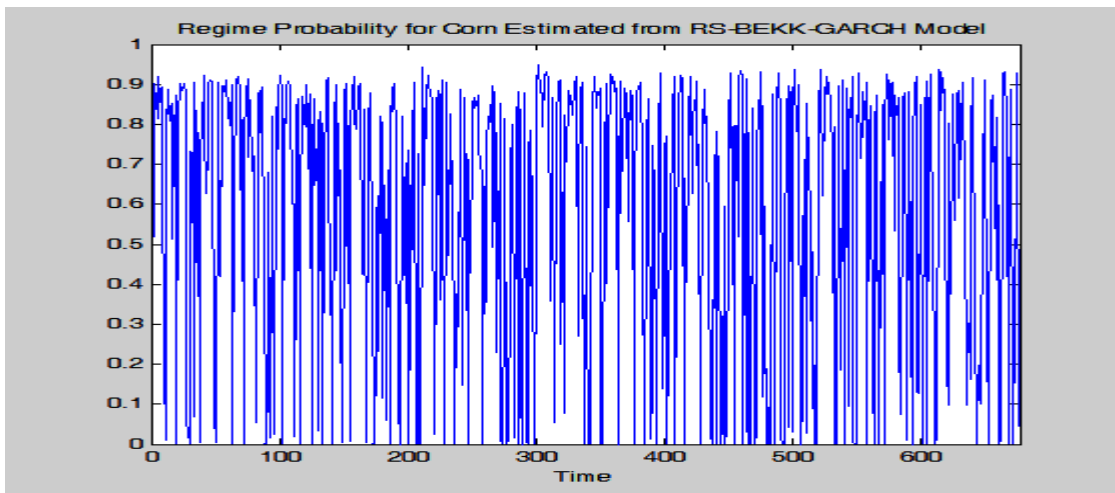


Figure 2
Regime probability of being in state 1 estimated from RS-BEKK model for corn

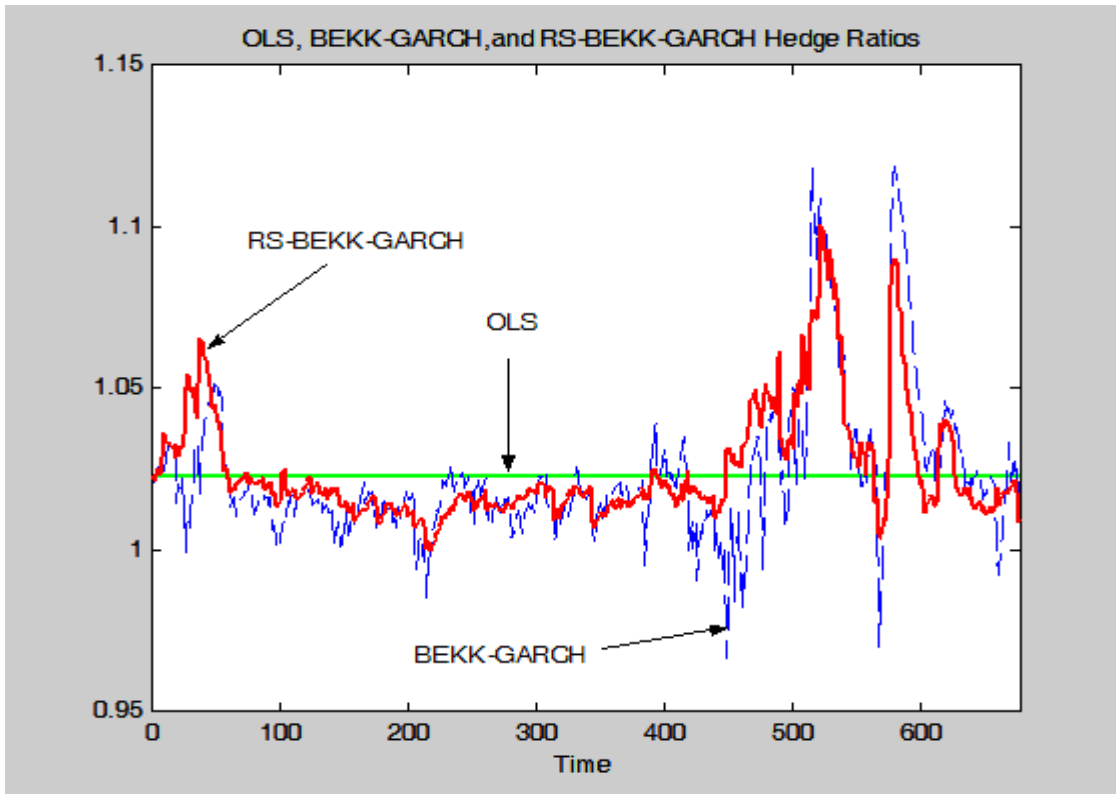


Figure 3
RS-BEKK, BEKK and OLS hedge ratios for nickel

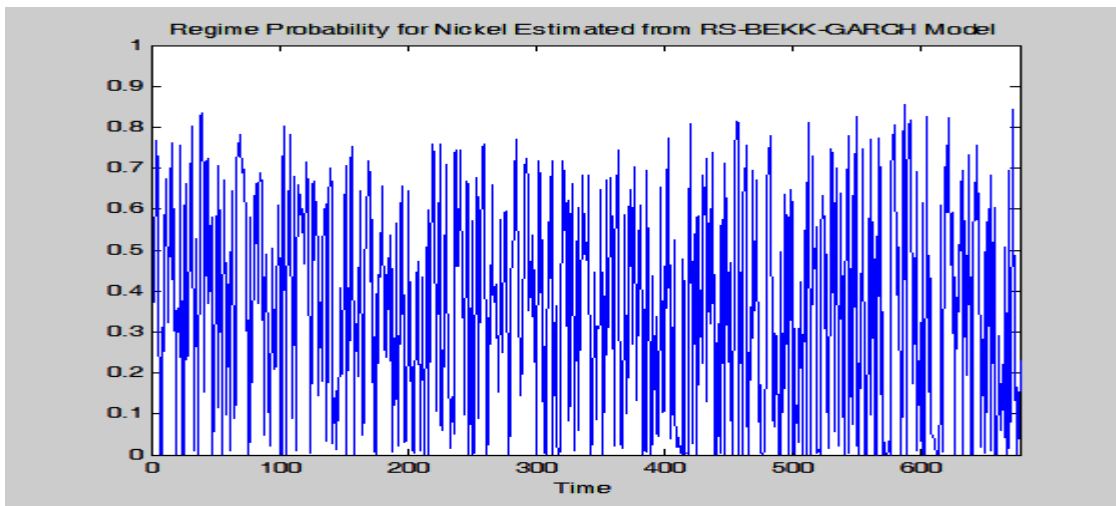


Figure 4
Regime probability of being in state 1 estimated from RS-BEKK model for nickel

Appendix A

To facilitate the explanation of the recombining method for RS-BEKK we replicate some figures from Gray's paper (1996) and compare the difference between our path-independent BEKK model from path dependent GARCH model and Gray's univariate path-independent GARCH model.

Figure A1 illustrates the evolution of conditional variances in a path-dependent GARCH model. Each conditional variance depends not just on the current regime, but on the entire past history of the process. The subscripts show the evolution of regimes. The term, $h_{2|1,2}^2$, for example, stands for the conditional variance at time 2, given that the process was in regimes 1 and 2, respectively, at times 1 and 2. Similarly, $e_{1|2}^2$ represents the square residual at time 1, given that the process was then in regime 2.

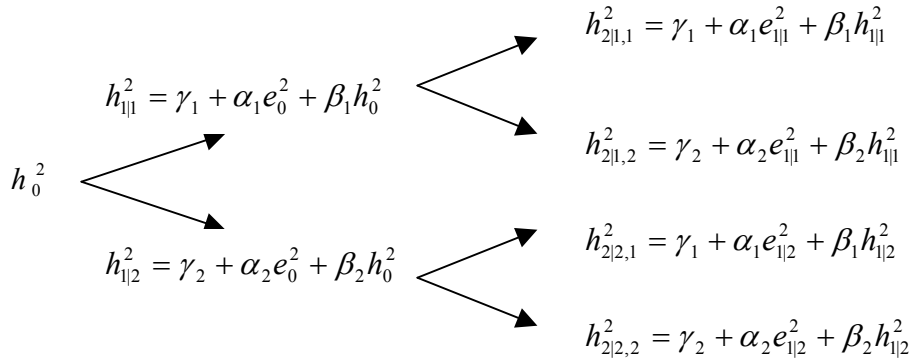


Figure A1 : Evolution of conditional variances in a univariate path-dependent GARCH model

Figure A2 illustrates the evolution of conditional variances in a univariate path-independent GARCH model. At each point in time, the conditional variance and residuals in each possible regime are recombined into a single conditional variance and residuals by taking expectation over the possible states in period 1. The conditional variance then depends only on the current regime, not on the entire past history of the process. The term $h_{2|1}^2$, for example, stands for the conditional variance at time 2, given that the process is then in regimes 1. $h_{1|1}^2$ and e_1 are the conditional variance and residual, respectively, after recombining at time 1.

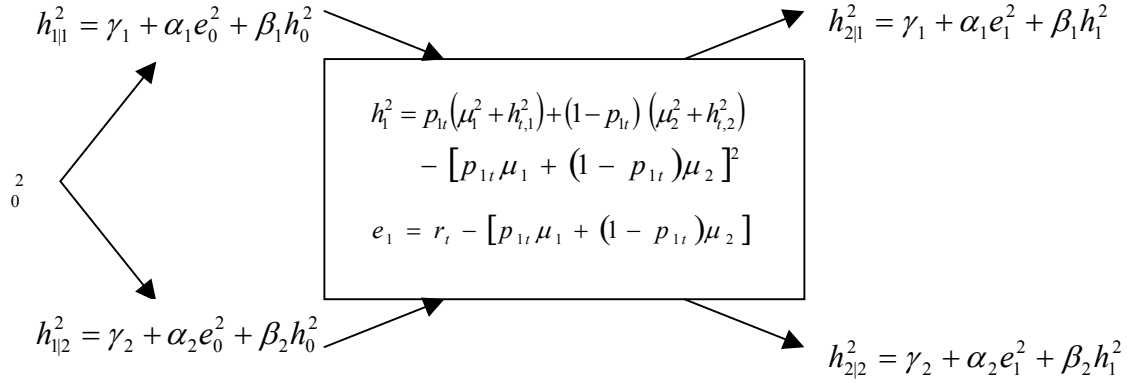


Figure A2 : Evolution of conditional variance in a univariate path-independent GARCH model

Figure A3 illustrates the evolution of conditional variances matrix in our path-independent RS-BEKK model. $\mathbf{H}_{2|1}$, for example, stands for the conditional variance-covariance matrix at time 2, given that the process is then in regimes 1. $h_{i,1}^2$ and $e_{i,1}$ are the conditional variance and residual, respectively, for asset $i \in \{c, f\}$ (cash and futures) after

recombining at time 1. $h_{cf,1}^2$ is the covariance of spot and futures returns after recombining at time 1. Γ , \mathbf{A} , and \mathbf{B} are 2 by 2 coefficient matrices that include for γ 's, α 's, and β 's, respectively (see equation 22). \mathbf{E} and \mathbf{H} are the residual matrix and variance matrix defined in equation (22).

At each point in time, the conditional variances and residuals in each possible regime are collapsed into a single conditional variance and residual by using Gray's recombining method for both spot and futures returns. Since our model is bivariate, we also need a recombining method for the covariance term. By taking the conditional expectation we can recombine the conditional covariance of spot and futures returns in each regime into a single conditional covariance, so that it depends only on the current regime, not on the entire past history of the process.

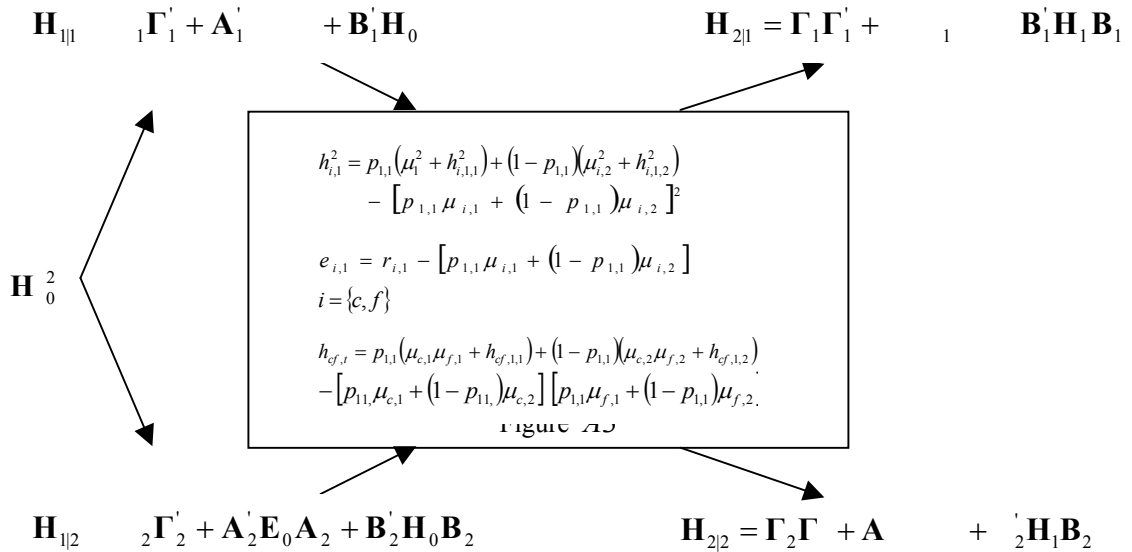


Figure A3: Evolution of conditional variance in a bivariate path-independent GARCH model

Appendix B

The recursive expression of the state probability p_{1t} of being in regime 1 at time t given all information up to time $t-1$ shown in equation (25) is proved in this appendix. It is a bivariate extension of Gray (1995, 1996).

According to a first-order Markov process, $p_{1t} = \Pr(s_t = 1 | \boldsymbol{\psi}_{t-1})$ depends only on the regime the process is in at time $t-1$. By conditioning on the regime at time $t-1$, we have

$$\begin{aligned} p_{1t} &= \Pr(s_t = 1 | \boldsymbol{\psi}_{t-1}) = \sum_{i=1}^2 \Pr(s_t = 1 | s_{t-1} = i) \Pr(s_{t-1} = i | \boldsymbol{\psi}_{t-1}) \\ &= P * \Pr(s_t = 1 | \boldsymbol{\psi}_{t-1}) + (1 - Q) * [1 - \Pr(s_t = 1 | \boldsymbol{\psi}_{t-1})], \end{aligned} \quad (\text{B1})$$

where $P = \Pr[s_t = 1 | s_{t-1} = 1]$, and $Q = \Pr[s_t = 2 | s_{t-1} = 2]$.

By using Bayes' Rule, $\Pr(s_t = 1 | \boldsymbol{\psi}_{t-1})$ can be written as a function of $\Pr(s_{t-1} = 1 | \boldsymbol{\psi}_{t-2})$:

$$\begin{aligned} \Pr(s_t = 1 | \boldsymbol{\psi}_{t-1}) &= \Pr(s_{t-1} = 1 | Y_{t-1}, \boldsymbol{\psi}_{t-2}) \\ &= \frac{f(\mathbf{R}_{t-1} | s_{t-1} = 1, \boldsymbol{\psi}_{t-2}) \Pr(s_{t-1} = 1 | \boldsymbol{\psi}_{t-2})}{f(Y_{t-1} | s_{t-1} = 1, \boldsymbol{\psi}_{t-2}) \Pr(s_{t-1} = 1 | \boldsymbol{\psi}_{t-2}) + f(\mathbf{R}_{t-1} | s_{t-1} = 2, \boldsymbol{\psi}_{t-2}) \Pr(s_{t-1} = 2 | \boldsymbol{\psi}_{t-2})} \end{aligned} \quad (\text{B2})$$

where $\mathbf{R}_t = [r_{c,t} \quad r_{f,t}]'$ is a vector of spot and futures returns at time t . Define f_{it} as:

$$f_{it} = f(\mathbf{R}_t | s_t = i, \boldsymbol{\psi}_{t-1}) = (2\pi)^{-1} |\mathbf{H}_{t,i}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \mathbf{e}_{t,i}' \mathbf{H}_{t,i}^{-1} \mathbf{e}_{t,i}\right\}, \quad i = 1, 2$$

and substitute (B2) into (B1), we can derived the recursive expression of the regime probability p_{1t} as

$$p_{1t} = P \left[\frac{f_{1t-1} p_{1t-1}}{f_{1t-1} p_{1t-1} + f_{2t-1} (1 - p_{1t-1})} \right] + (1 - Q) \left[\frac{f_{2t-1} (1 - p_{1t-1})}{f_{1t-1} p_{1t-1} + f_{2t-1} (1 - p_{1t-1})} \right], \quad (\text{B3})$$

where B3 is the same as equation (25) in the text.