

Testing Cointegration Rank in Large Systems

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Abstract

In this paper we investigate the possibility of the application of subsampling procedure for testing cointegration relations in large multivariate systems. The subsampling technique is applied to overcome the difficulty of nonstandard distribution and nuisance parameters in testing for cointegration rank without an explicitly formulated structural model. The contribution in this paper is twofold: theoretically this paper shows that the subsampling testing procedure is consistent and asymptotically most powerful; practically this paper demonstrates that the subsampling procedure can be applied to determine the cointegration rank in large scale models, where the standard procedures hits already its limit. Especially for the cases of few stochastic trends in a system, the subsampling procedure shows robust and reliable results.

KEY WORDS: Cointegration, Large Systems, Nonparametric Tests, Subsampling, PPP

1 Introduction

Over last decade considerable attention has been paid in empirical economics to testing for the existence of long-run relations, using cointegration techniques. There have been two main approaches: the two step residual-based procedure for testing either the null of no cointegration¹ or the null of cointegration² and the system-based reduced rank regression approach³. The residual-based approach is applicable to analyzing single cointegration relations between variables⁴. The system-based approach is applicable to small systems with a few variables due to its data intensive specifications of the error correction models⁵.

Recently some non-parametric and semi parametric approaches are developed to test the cointegration relations. Bierens (1997) proposes a testing procedure that is free from the specification of the data-generating process. Breitung (2002) proposes a testing procedure that is invariant against log transformation which is often used in the empirical applications. Boswijk and Lucas (2002) aims at rising the power of testing procedure. Following this line of research we propose a nonparametric subsampling approach to testing for cointegration relations, targeting at handling cointegration analysis in large systems with moderate demand for data.

Large systems have always been the interest of research. For instance, finding the number of independent stochastic trends in stock prices or bond prices that all behavior like stochastic trends will help to gain more insight into the driving forces of the financial markets, and knowing the number of independent stochastic trends among main economic indicators of an economy will help to understand the structure of the economy. But, the present standard procedure of multivariate cointegration analysis - the Johansen procedure can not applied to large systems. The basic reason is that the Johansen procedure requires a completely specified vector error model(ECM). In case of large scale systems this will certainly quickly use up degree of freedom for practically available economic data. Furthermore, the large number of parameters in large VECM make the statistical inference very unreliable.

The subsampling procedure for testing cointegration rank turns out in many cases to be simple, robust and has correct size, especially when the system is large and the number of independent stochastic trend is small. The plan of

¹See Engle and Granger (1987), Phillips and Quliaris (1990)

²See Kwiatkowski, Phillips, Schmidt, and Shin (1992)

³Johansen (1991) and Johansen (1995)

⁴For problems of the application of this approach to many variables see Hamilton (1994)

⁵Empirical systems analyzed by this approach usually do not exceed the dimensionality of 8. Even the most used softwares provide only critical values of Johansen procedure up to the dimensionality of 12.

this paper is as follows: Section 2 sets out the underlying model and motivates the basic idea of the testing procedure. Section 3 and Section 4 describe the testing procedure and consider the properties of the test statistics. Section 5 presents some simulation results. Section 6 describes an empirical application. Section 7 provides some concluding remarks. Technical details are included in the Appendix.

2 Assumption of DGP and Motivation of the Test

Let the $n \times 1$ time series y_t be specified as follows:

$$y_t = C\xi_t + u_t, \quad (2.1)$$

where u_t is $n \times 1$ stationary time series

$$u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

with $\sum_{j=0}^{\infty} |\psi_j| < \infty$; and ξ_t is $g \times 1$ random walk ⁶

$$\xi_t = \sum_{s=1}^t \epsilon_{2,t,s}$$

⁶This model can accommodate the model:

$$\xi_t = \sum_{s=1}^t u_{2,t,s}$$

with $u_{2,t} = \sum_{j=0}^{\infty} \psi_{2,j} \epsilon_{2,t-j}$, $\sum_{j=0}^{\infty} j |\psi_{2,j}| < \infty$ by using the Beveridge-Nelson decomposition, see Hamilton (1994) P.504. In this context the formulation (2.1) encompasses the error correction model. To illustrate this we consider following error correction representation of a cointegrated system:

$$\Delta y_t = \beta \alpha' y_{t-1} + \pi \Delta y_{t-1} + \epsilon_t. \quad (2.2)$$

This model can be reformulated as follows:

$$\begin{aligned} y_t &= (I + \beta \alpha') y_{t-1} + \pi \Delta y_{t-1} + \epsilon_t \\ &= (\alpha^*, \alpha_{\perp}^*) \begin{pmatrix} \alpha' \\ \alpha_{\perp}' \end{pmatrix} y_{t-1} + \beta \alpha' y_{t-1} + \pi \Delta y_{t-1} + \epsilon_t \\ &= \alpha_{\perp}^* \alpha_{\perp}' y_{t-1} + (\alpha^* + \beta) \alpha' y_{t-1} + \pi \Delta y_{t-1} + \epsilon_t \\ &= C \xi_t + u_t \end{aligned}$$

where $C = \alpha_{\perp}^*$ is a $n \times g$ matrix; $\xi_t = \alpha_{\perp}' y_{t-1}$ is $g \times 1$ stochastic trend; and $u_t = (\alpha^* + \beta) \alpha' y_{t-1} + \pi \Delta y_{t-1} + \epsilon_t$ is $n \times 1$ stationary series.

where $(\epsilon_t, \epsilon_{2,t})$ are jointly gauss-distributed with $\epsilon_t \sim N(0, I_n)$, $\epsilon_{2,t} \sim N(0, I_g)$ and $\Omega = Cov(\epsilon_t, \epsilon'_{2,t})$. C is a $n \times g$ matrix with rank g . This model has g independent stochastic trends. Let β be $n \times h$ matrix ($h := n - g$) such that $\beta' C = 0$. β is called *(one) basis of cointegration vectors* because

$$\beta' y_t = \beta' u_t \quad (2.3)$$

is stationary. Thus, this system has h cointegration relations⁷. The correspondence of the cointegration representation (2.1) and the standard representations, for example, the Phillips's triangular representation, the Stock-Watson common trends representation and the error correction representation can be seen in Hamilton (1994) Chap.18. We choose the representation of (2.1) and (2.3) to pave the way for a test without specifying a structural model for the DGP.

(2.1) implies that cointegration is basically a phenomena of multicollinearity up to a I(0) residuals. Hence the rank of the regression coefficient matrix of the regression of Δy_t on y_{t-1} should reveal the number independent stochastic trends. Difficulty in finding a decision rule to determine the number of independent stochastic trends lies in the description of the distribution of the statistics on the rank of the coefficient matrix, because it may not have a standard distribution and may depend on nuisance parameters. This issue is fully discussed in Stock and Watson (1988), where they construct the statistical tests by looking at the corrected first order series correlation matrix of the observed data, which yields tabulable standard distributions. Johansen (1991) and Johansen (1995) take the maximum likelihood approach and construct tabulable statistical tests by controlling for correlation of the I(0) components through a completely specified structural model for the DGP.

Alternatively, we apply subsampling method to construct a subsampling confidence interval for the test, such that we avoid the problem of the description of the distribution of the test statistic by drawing from the "true" distribution- the DGP.

The first step of the procedure is to regress Δy_t on y_{t-1} by OLS like follows:

$$\Delta y_t = \Pi y_{t-1} + e_t \quad (2.4)$$

We get the OLS estimator $\hat{\Pi}$:

$$\hat{\Pi}' = \left(\sum_{t=1}^T y_{t-1} y'_{t-1} \right)^{-1} \left(\sum_{t=1}^T y_{t-1} \Delta y'_t \right) \quad (2.5)$$

⁷Following the general characterization of cointegrating vector in Hamilton (1994) P. 547, (2.1) and (2.3) consist a cointegrating system.

This regression equation differs from the error correction model of Johansen procedure only in that it omits the specification of the short run dynamics. Owing to the super consistence of LS in regression with I(1) variables, we would nevertheless expect that the rank of $\hat{\Pi}$ would converge to the cointegration rank $n - g$. Thus the information about the rank of $\hat{\Pi}$, or equivalently, the number of zero eigenvalues of $\hat{\Pi}$ can be used to test the cointegration rank.

The advantage of Johansen procedure is that through controlling for the short run dynamics it achieves a test statistic based on the rank of the regression matrix of Δy_t on y_{t-1} ⁸ that is free from nuisance parameter and hence tables of critical values for the test statistics can be calculated. However, the full specification of the short run dynamics will quickly use up the degree of freedom for large systems, such that Johansen procedure is only practically applicable for small systems due to the availability of data for empirical research in the real world.

We sacrifice the specification of the short run dynamics for the ability to handle large cointegration systems at the same availability of data and hope that the super consistence of $\hat{\Pi}$ will nevertheless provides a good estimate for the calculation of the rank of Π . The difficulty is that the resulting test statistic of the rank of $\hat{\Pi}$ will be nonstandard and depends on nuisance parameters. We overcome this problem by using subsampling method that is immune against nonstandard distribution and the presence of nuisance parameters. This is the motivation behind the test procedure.

3 Property of the LS Regression

The last section suggests that we could infer the number of the cointegration relations from the rank of the estimated matrix $\hat{\Pi}$

$$\hat{\Pi} = \left(\sum_{t=0}^{T-1} y_t y_t' \right)^{-1} \left(\sum_{t=0}^{T-1} y_t \Delta y_{t+1}' \right). \quad (3.6)$$

where $\Delta y_t = C\epsilon_{2,t} + u_t - u_{t-1} =: v_t$. In this section the distribution of the eigenvalues of $\hat{\Pi}$ will be derived. We will see that the eigenvalues can be separated into two groups which have different order of convergence as $T \rightarrow \infty$; one group corresponds to the cointegration space and the other group corresponds to the space of stochastic trends. Later we will take advantage of this fact to construct our test statistics.

Let β_{\perp} be a $n \times g$ matrix including a basis of the complement subspace to β . Let $B = (\beta, \beta_{\perp})$.

⁸The the matrix of canonical correlation coefficient has the same rank as the matrix $\hat{\Pi}$.

First we look at the following terms

$$\begin{aligned}
B' \sum_{t=0}^{T-1} y_t y_t' B &= \begin{pmatrix} \beta' \\ \beta_\perp' \end{pmatrix} \sum_{t=0}^{T-1} (C \xi_t \xi_t' C' + u_t \xi_t' C' + C \xi_t u_t' + u_t u_t') \begin{pmatrix} \beta & \beta_\perp \end{pmatrix} \\
&= \begin{pmatrix} \beta' (\sum_{t=0}^{T-1} u_t u_t') \beta & \beta' (\sum_{t=0}^{T-1} u_t \xi_t' C' + \sum_{t=0}^{T-1} u_t u_t') \beta_\perp \\ \beta_\perp' (\sum_{t=0}^{T-1} C \xi_t u_t' \beta + \sum_{t=0}^{T-1} u_t u_t') \beta & \beta_\perp' \sum_{t=0}^{T-1} (C \xi_t + u_t) (\xi_t' C' + u_t') \beta_\perp \end{pmatrix} \\
&=: \begin{pmatrix} G_{1,T} & G_{2,T} \\ G_{3,T} & G_{4,T} \end{pmatrix}
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
B' \sum_{t=0}^{T-1} y_t \Delta y_{t+1}' &= \begin{pmatrix} \beta' \\ \beta_\perp' \end{pmatrix} \sum_{t=0}^{T-1} (C \xi_t v_{t+1}' + u_t v_{t+1}') \\
&= \begin{pmatrix} \beta \sum_{t=0}^{T-1} u_t v_{t+1}' \\ \beta_\perp' \sum_{t=0}^{T-1} (C \xi_t + u_t) v_{t+1}' \end{pmatrix} \\
&=: \begin{pmatrix} G_{5,T} \\ G_{6,T} \end{pmatrix}.
\end{aligned} \tag{3.8}$$

Before we discuss the limit distributions of the matrices above we define $w_t = \sum_{s=0}^{\infty} \phi_{1,s} \epsilon_{t-s} + \sum_{s=0}^{\infty} \phi_{2,s} \epsilon_{2,t-s}$, where w_t is $n \times 1$ vector, $\phi_{1,s}$ are $n \times n$ matrices and $\phi_{2,s}$ are $n \times g$ matrices. Recall that the $n \times g$ matrix Ω is the covariance matrix of ϵ_t and $\epsilon_{2,t}$. Let \xrightarrow{P} denote the convergence in probability and \xrightarrow{D} denote the convergence in distribution-

Lemma 3.1 (i)

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} u_t w_t' \xrightarrow{P} \sum_{j=0}^{\infty} \psi_j \phi_{1,j}' + \sum_{j=0}^{\infty} \psi_j \Omega \phi_{2,j}'$$

(ii)

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \xi_t w_t' = \Omega' \phi_1(1)' + \phi_2(1)' + \int_0^1 W_{2,s} dW_s' \phi_1(1)' + \int_0^1 W_{2,s} dW_{2,s}' \phi_2(1)',$$

where $(W_s, W_{2,s})$ are jointly Brownian motion with the same correlation as $(\epsilon_s, \epsilon_{2,s})$ and $\phi_i(1) = \sum_{j=0}^{\infty} \phi_{i,j}$ for $i = 1, 2$.

(iii)

$$\lim_{T \rightarrow \infty} \frac{1}{T^2} \sum_{t=0}^{T-1} \xi_t \xi_t' = \int_0^1 W_{2,s} W_{2,s}' ds.$$

Proof

The statements (i), (ii) and (iii) follow the results of (f), (h) and (b) of Lemma 1 in Sims, Stock, and Watson (1990) P.121. \square

Here we introduce two notations:

$$G_T = O_C(T^\alpha) \quad \text{iff} \quad \lim_{T \rightarrow \infty} \frac{G_T}{T^\alpha} \xrightarrow{P} g \quad (g \text{ is a constant and } g \neq 0),$$

$$G_T = O_D(T^\alpha) \quad \text{iff} \quad \lim_{T \rightarrow \infty} \frac{G_T}{T^\alpha} \xrightarrow{D} G \quad (G \text{ is a random variable and } G \neq 0).$$

Remark $O_C(T^\alpha)$ can be seen as a special case of $O_D(T^\alpha)$ where $O_C(T^\alpha)/T^\alpha$ converges to a degenerated distribution (constant). Sometimes we do not distinguish these two cases if the difference is not relevant.

Using Lemma 3.1 we have

Property 3.2

$$\begin{aligned} G_{1,T} &= O_C(T) & G_{2,T} &= O_D(T) \\ G_{3,T} &= O_D(T) & G_{4,T} &= O_D(T^2) \\ G_{5,T} &= O_C(T) & G_{6,T} &= O_D(T) \end{aligned}$$

\square

Let G_i be the limit (as a distribution) of $G_{i,T}$ for $i = 1, \dots, 6$, that means

$$G_i := \lim_{T \rightarrow \infty} \frac{G_{i,T}}{T^{\alpha_i}} \quad \text{in distribution.}$$

Lemma 3.3 $G_{i,T} = O_D(T^{\alpha_i})$ for $i = 1, \dots, 6$ are defined in (3.7) and (3.8). G_i are the limit as described above. Then it holds

(i) For $G_{i,T}$ and G_i invertible,

$$G_{i,T}^{-1} = O_D(T^{-\alpha_i})$$

(ii) For $G_{i,T}, G_{j,T}$ such that $G_i \neq 0, G_j \neq 0$ and that $G_{i,T} \cdot G_{j,T}$ and $G_i \cdot G_j$ are well-defined matrix multiplication and $G_i \cdot G_j \neq 0$, we have then

$$G_{i,T} \cdot G_{j,T} = O_D(T^{\alpha_i + \alpha_j}).$$

(iii) If $\alpha_i > \alpha_j$, then

$$G_{i,T} + G_{j,T} = O_D(T^{\alpha_i}).$$

Proof See Appendix.

Remark The results (i), (ii) and (iii) hold for $G_{i,T}$ as defined above, because these sequences $G_{i,T}$ are continuous functionals of the white noise $(\epsilon_t, \epsilon_{2,t})$

so that we can apply the continuous mapping theorem to them⁹. As the result, the product of two sequences converges to the product of the limits in distribution as (ii) in the Lemma above. Similarly it holds for inversion and summation as in (i) and (iii).

Let

$$\begin{pmatrix} G_{7,T} \\ G_{8,T} \end{pmatrix} := \left(B' \sum_{t=0}^{T-1} y_t y_t' B \right)^{-1} B' \sum_{t=0}^{T-1} y_t v_{t+1}' = \begin{pmatrix} G_{1,T} & G_{2,T} \\ G_{3,T} & G_{4,T} \end{pmatrix}^{-1} \begin{pmatrix} G_{5,T} \\ G_{6,T} \end{pmatrix} = B^{-1} \hat{\Pi}.$$

We denote particularly G_1 and G_5 as g_1 and g_5 in small capital to emphasize that the limits are constants instead of (non-degenerated) random variables. Then we have:

Lemma 3.4

$$\begin{aligned} \lim_{T \rightarrow \infty} G_{7,T} &\xrightarrow{P} g_1^{-1} g_5 =: g_7 \\ \lim_{T \rightarrow \infty} T G_{8,T} &\xrightarrow{D} -g_5 G_4^{-1} G_3 g_1 + G_6 G_4 =: G_8. \end{aligned}$$

Proof See Appendix.

Following Lemma 3.4 $T\hat{\Pi}$ behaves asymptotically like follows:

$$T\hat{\Pi} = B \begin{pmatrix} T G_{7,T} \\ T G_{8,T} \end{pmatrix} \sim B \begin{pmatrix} T g_7 \\ G_8 \end{pmatrix}. \quad (3.9)$$

Now we are interested in the distribution of the *eigenvalues* of $T\hat{\Pi}$. We present the following theorem:

Theorem 3.5 *Let $\lambda_{1,T}, \dots, \lambda_{n,T}$ be the eigenvalues of $T\hat{\Pi}$. Then there are h eigenvalues for $i = 1, \dots, h$ such that*

$$\lim_{T \rightarrow \infty} \frac{\lambda_{i,T}}{T} \xrightarrow{P} \lambda_i \quad \text{constants} \quad (3.10)$$

and the rest g eigenvalues for $i = h + 1, \dots, n$

$$\lim_{T \rightarrow \infty} \lambda_{i,T} \xrightarrow{D} \lambda_i \quad \text{distributions} \quad (3.11)$$

.

Proof See Appendix.

⁹See Hamilton (1994) P.482.

Corollary 3.6

$$\lim_{T \rightarrow \infty} \frac{|\lambda_{i,T}|}{T} \xrightarrow{P} |\lambda_i| \quad \text{constants} \quad (3.12)$$

and the rest g eigenvalues for $i = h + 1, \dots, n$

$$\lim_{T \rightarrow \infty} |\lambda_{i,T}| \xrightarrow{D} |\lambda_i| \quad \text{distributions} \quad (3.13)$$

4 The Test Procedure

The hypothesis of the existence of g independent stochastic trends or equivalently h cointegration relations is that the g in absolute value smallest eigenvalues of Π are zero.

$$H_0 : \lambda_i = 0, \quad i = 1, 2, \dots, g \quad (4.14)$$

where λ_i are Eigenvalues of Π in increasing order.

(3.12) and (3.13) tell us if we were able to construct a $1 - \alpha$ percent confidence interval based on the distribution $D_i, i = 1, 2, \dots, g$, then $1 - \alpha$ percent of the g smallest eigenvalues of the matrix $T\hat{\Pi}$ will lie in the corresponding confidence interval, while the other $n - g$ eigenvalues will asymptotically escape from any empirical confidence interval constructed by using subsamples. This difference between the divergence of the h largest eigenvalues of $T\hat{\Pi}$ and the convergence of the g smallest eigenvalues yields a sharp separation between these two groups of eigenvalues. In this sense this test of the cointegration rank is asymptotically most powerful.

Using Theorem 4.2.1 of Politis, Romano, and Wolf (1999), and the results of Theorem 3.5 we can get asymptotic valid test results.

Following is the result from Politis et al. (1999):

Let $\hat{\theta}_T = \hat{\theta}(Y_1, Y_2, \dots, Y_T)$ be an estimator of $\theta(P) \in \mathbb{R}$, the parameter of interest. Let $\hat{\theta}_{T,b,t} = \hat{\theta}_b(Y_t, Y_{t+1}, \dots, Y_{t+b-1})$ the estimator of $\theta(P)$ based on the subsample $\{Y_t, \dots, Y_{t+b-1}\}$. Define $J_{b,t}(P)$ be the sample distribution of $\tau_b(\hat{\theta}_{n,b,t} - \theta(P))$, where τ_b is an appropriate normalizing constant. Also, define the corresponding cumulative distribution function:

$$J_{b,t}(x, P) = \text{Prob}_P\{\tau_b(\hat{\theta}_{n,b,t} - \theta(P)) \leq x\}$$

For notation convenience, let $J_T(P) = J_{1,T}(P)$ and $J_T(\cdot, x) = J_{1,T}(\cdot, P)$. Denote the empirical distribution function as follows:

$$L_{T,b}(x) = \frac{1}{T - b + 1} \sum_{t=1}^{T-b+1} 1\{\tau_b(\hat{\theta}_{n,b,t} - \hat{\theta}_T) \leq x\}.$$

Essentially the only assumption that will be needed to consistently estimate $J_T(P)$ is the following:

Assumption 4.1 *There exists a limiting law $J(P)$ such that*

- *i. $J_T(P)$ converges weakly to $J(P)$ as $T \rightarrow \infty$.*
- *ii. For every continuity point x of $J(P)$ and for any sequences T, b with $T, b \rightarrow \infty$ and $b/T \rightarrow 0$, we have $\frac{1}{T-b+1} \sum_{t=1}^{T-b+1} J_{b,t}(x, P) \rightarrow J(x, P)$*

Theorem 4.2 (subsampling) *Assume Assumption 4.1 and that $\tau_b/\tau_T \rightarrow 0$, $b/T \rightarrow 0$, and $b \rightarrow \infty$ as $T \rightarrow \infty$. Also that $\alpha_X(m) \rightarrow 0$ as $m \rightarrow \infty$.*

- *i. If x is a continuity point of $J(., P)$, then*

$$L_{b,T}(x) \xrightarrow{P} J(x, P) \quad (4.15)$$

- *ii. If $J(., P)$ is continuous, then*

$$\sup_x |L_{b,T}(x) - J(x, P)| \xrightarrow{P} 0 \quad (4.16)$$

- *iii. For $\alpha \in (0, 1)$ let*

$$c_{T,b}(1 - \alpha) = \inf\{x : L_{T,b}(x) \geq 1 - \alpha\}. \quad (4.17)$$

Define:

$$c(1 - \alpha, P) = \inf\{x : J(x, P) \geq 1 - \alpha\}. \quad (4.18)$$

If $J(., P)$ is continuous at $c(1 - \alpha, P)$, then

$$\text{Prob}_P\{\tau_T[\hat{\theta}_T - \theta(P)] \leq c_{T,b}(1 - \alpha)\} \xrightarrow{P} 1 - \alpha \quad \text{as } T \rightarrow \infty. \quad (4.19)$$

Thus, the asymptotic coverage probability under P of the interval $I_1 = [\hat{\theta}_T - \tau_T^{-1}c_{T,b}(1 - \alpha), \infty)$ is the nominal level $1 - \alpha$.

Proof: see Politis et al. (1999)

Comments:

For the application of the theorem above to testing for cointegration rank, the limiting law are the weakly convergence in (3.13). Because every summand in (ii) of Assumption 4.1 has the same limiting distribution for $b \rightarrow \infty$, the

average has the same limiting distribution. Therefore, we can apply the subsampling procedure to get an asymptotically valid test for the cointegration rank.

Under H_0 of g independent stochastic trends, up to g -th smallest eigenvalue of the test statistic of the whole sample $T\hat{\Pi}$ will lie in the corresponding empirical confidence interval constructed by subsampling with probability $1 - \alpha$ while the other $(n - g)$ eigenvalues of the test statistic of the whole sample will lie beyond the corresponding "subsampling confidence interval".

The testing procedure:

- Calculation of $|T\Lambda(\hat{\Pi})|$ as the whole sample test statistic for each eigenvalues, where $\Lambda(\hat{\Pi})$ is the diagonal matrix of the eigenvalues of $\hat{\Pi}$.
- Choice of a proper subsample size b_T ¹⁰
- Calculation of $T - b + 1$ subsample test statistics $|b_T\Lambda(\hat{\Pi}_{b_T,t})|$ ($t = b, b + 1, \dots, T$) for each norm of eigenvalues of $T\hat{\Pi}$
- Calculation of the empirical distribution function based on the subsample test statistics and calculation of the empirical confidence intervals at a given confidence level α for each norm of eigenvalues of $T\hat{\Pi}$.
- Check of the whole sample test statistic with the corresponding confidence interval and conclude the test result.

Remark 1

We can also calculate the sample canonical correlation coefficient as the test statistic. For this alternative way we have following results. Let Π^n be the matrix to calculate the sample canonical coefficient between ΔY_t and Y_{t-1} .

$$\Pi^n = \left(\sum_{t=0}^{T-1} \Delta y_{t+1} \Delta y'_{t+1} \right)^{-1} \left(\sum_{t=0}^{T-1} \Delta y_{t+1} y'_t \right) \left(\sum_{t=0}^{T-1} y_t y'_t \right)^{-1} \left(\sum_{t=0}^{T-1} y_t \Delta y'_{t+1} \right).$$

We have:

$$\begin{aligned} & T\Pi^n \\ &= T \left(\sum_{t=0}^{T-1} \Delta y_{t+1} \Delta y'_{t+1} \right)^{-1} \left(\sum_{t=0}^{T-1} \Delta y_{t+1} y'_t \right) B B^{-1} \left(\sum_{t=0}^{T-1} y_t y'_t \right)^{-1} B^{-1} B \left(\sum_{t=0}^{T-1} y_t \Delta y'_{t+1} \right) \\ &= \begin{pmatrix} O_c(T) & O_c(T) \\ O_c(T) & O_c(T) \end{pmatrix}^{-1} \begin{pmatrix} O_C(T) & O_D(T) \end{pmatrix} \begin{pmatrix} G_{1,T} & G_{2,T} \\ G_{3,T} & G_{4,T} \end{pmatrix} \begin{pmatrix} G_{5,T} \\ G_{6,T} \end{pmatrix} \\ &\xrightarrow{L} (Tg_9, G_{10}) \end{aligned}$$

¹⁰Theoretically any subsample block sizes satisfying the condition $b \rightarrow \infty$ and $b/T \rightarrow 0$ as $T \rightarrow \infty$ will have the same asymptotical result. However the choice of an optimal is sensitive practical issue, because the test results may depend on the choice of the block size. See Politis et al. (1999) and Choi (2003) for detailed discussion.

The convergency in distribution in the last line above follows from the results of Lemma 4.3 and the Slutsky theorem¹¹. Applying the results of Theorem 3.5 we can also apply subsampling procedure to the sample canonical coefficient to test the cointegration rank. One advantage of using canonical correlation coefficient instead of Π is that we do not need to care about the problem of complex eigenvalues, because the matrix used to calculate the canonical correlation coefficients is positive definite, while $\hat{\Pi}$ may contains complex eigenvalues.

Remark 2

In the presentation above we have assumed that there is no drift involved in the stochastic trends. However, the subsampling result can still apply for the cases with drifts, if $g > 1$. In the presence of drifts in the stochastic trends, only the smallest eigenvalue of $\hat{\Pi}$ will converge at the rate $T^{-\frac{3}{2}}$ to zero under null. The convergence rate of other $g - 1$ eigenvalues that corresponds to the space independent stochastic trends are T^{-1} , and the convergence rate of the remaining $n - g$ eigenvalues that corresponds to the cointegration space are $T^{-\frac{1}{2}}$ ¹². For the null of $g = 1$ with drift we use the scaling factor $T^{\frac{3}{2}}$ to test for cointegration. Furthermore, we can apply different the scaling factor $T^{\frac{3}{2}}$ or T to the smallest eigenvalue to test the presence of drift in the stochastic trends.

5 Simulation Studies

Subsampling procedure is an asymptotically valid test. For the application of this procedure to empirical research the performance of the this procedure in limited sample size is relevant. Therefore we conduct some Monte Carlo simulation to assess the small sample property of the subsampling procedure. We would like to know how well the subsampling performs, when (1) the dimension of the DGP varies, and (2) the coefficients of autocorrelation in u_t increase.

The DGP for the simulation studies is:

$$y_t = C\xi_t + u_t \tag{5.20}$$

¹¹To apply Slutsky theorem we need in fact stronger condition, namely the product of convergence in probability and convergence in distribution. We have here the product of convergence in distribution only. However, following Föllmer (1981), the convergence of unit root processes to functionals Brownian motion has a version of strong convergence and hence forth convergence in probability. Therefore we can apply Slutsky theorem in our cases.

¹²See Sims et al. (1990) and Hamilton (1994) P.555 for more details.

with

$$\xi_t = \sum_{\tau=1}^t e_{\tau} \quad (5.21)$$

$$e_t = \sum_{\tau=0}^4 \rho_{e\tau} \epsilon_{t-\tau}, \quad \epsilon_{et} \sim i.i.d.N(0, I_g) \quad (5.22)$$

and

$$u_t = \sum_{\tau=0}^4 \rho_{\tau} \epsilon_{t-\tau}, \quad \epsilon_t \sim i.i.d.N(0, I_n) \quad (5.23)$$

Table 1 presents the results of some simulation studies of the subsampling procedure. The numbers in the table are the relative frequency on non-rejection of the H_0 of zero eigenvalues calculated through 100 replications.

No.	n	g_0	T	b	$H_0 : g = g_0 - 1$	$H_0 : g = g_0$	$H_0 : g = g_0 + 1$
1	15	5	200	90	0.86	0.88	0.00
2	20	5	200	90	0.82	0.85	0.00
3	30	5	200	90	0.78	0.84	0.00
4	40	5	200	90	0.81	0.79	0.00
5	60	5	200	90	0.74	0.83	0.00

Table 1: *Estimated coverage probability of various nominal 90% subsampling confidence intervals. n is the dimension of the system. g_0 is the number of independent stochastic trends of the DGP. g is the hypothetical numbers of independent stochastic trends under test. T is the total sample size. b is the used subsample size. $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0, \rho_{e1} = \rho_{e2} = \rho_{e3} = \rho_{e4} = 0$.*

Table 1 shows that the subsampling procedure provides satisfactory results even at moderate number of observations. Next we look at the impact of the presence of autocorrelation in u_t on the subsampling results. In terms of error correction model this can be equivalently formulated as the impact of short run dynamics on the subsampling results. Table 2 shows the results of some simulation studies. The numbers in the table are the relative frequency on non-rejection of the H_0 of zero eigenvalues of Π calculated through 100 replications.

No.	n	g_0	T	b	$H_0 : g_0 - 1$	$H_0 : g_0$	$H_0 : g_0 + 1$	ρ_1	ρ_2	ρ_3	ρ_4
1	15	5	200	110	0.81	0.85	0.00	0.3	0.1	0.1	0
2	20	5	200	110	0.84	0.84	0.00	0.3	0.1	0.1	0
3	30	5	200	110	0.82	0.81	0.00	0.3	0.1	0.1	0
4	40	5	200	110	0.75	0.65	0.00	0.3	0.1	0.1	0
5	50	5	200	110	0.75	0.49	0.00	0.3	0.1	0.1	0
6	60	5	200	110	0.62	0.41	0.00	0.3	0.1	0.1	0
7	15	5	300	130	0.92	0.85	0.00	0.4	0.2	0.1	0
8	20	5	300	130	0.86	0.81	0.00	0.4	0.2	0.1	0
9	30	5	300	130	0.85	0.80	0.00	0.4	0.3	0.1	0
10	40	5	300	150	0.85	0.75	0.00	0.4	0.2	0.1	0
11	50	5	400	150	0.81	0.68	0.00	0.4	0.3	0.1	0
12	60	5	400	150	0.79	0.58	0.00	0.4	0.4	0.1	0

Table 2: *Estimated coverage probability of various nominal 90% subsampling confidence intervals. n is the dimension of the systems. g_0 is the number of the independent stochastic trends. H_0 under test are $g = g_0 - 1, g = g_0$ and $g = g_0 + 1$ respectively. T is the total sample size. b is the subsample size. $\rho_{e1} = \rho_{e2} = \rho_{e3} = \rho_{e4} = 0$.*

These simulation results show that the subsampling is a very promising method to handel the test of cointegration rank in large scale models¹³. In most cases the test will have correct size¹⁴. Obviously, the power of the test is very high, i.e. in the simulation the testing procedure did not falsely accept any stochastic trend. This is due to the sharp contrast between the asymptotical diverging property of the h largest eigenvalues and the converging property of the g smallest eigenvalues. To illustrate this point we look at a simulation case with following parameters: $T = 200, n = 16, g = 5, b \in (50, 100), \rho_1 = \rho_2 = \rho_3 = \rho_4 = 0, \rho_{e1} = \rho_{e2} = \rho_{e3} = \rho_{e4} = 0$.

¹³The standard Johansen procedure will fail to apply due to its data demanding formulation, i.a. in case of a system with 50 variables, an error correction model with 4 lags will use up all the degree of freedoms for 200 observations.

¹⁴The size downwards distortion becomes significant when the dimension of the system become large. According to Politis et al. (1999) Chapter 9, calibration method can be used to adjust the nominal coverage frequency to get a correct size. We show the subsampling coverage frequency to document this nominal downwards distortion in case of limited sample size.

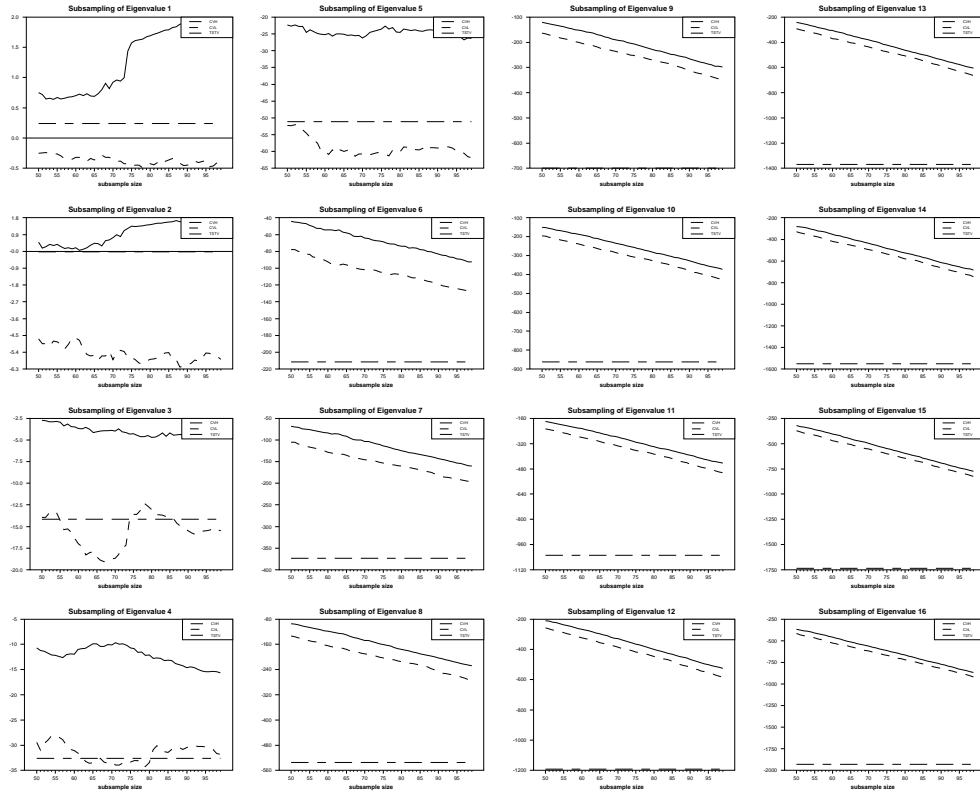


Figure 1: *Subsampling confidence interval without autocorrelation*

The 4 graphs in the first column and first graph in the second column show the subsampling confidence intervals for the 5 smallest eigenvalues that correspond to 5 independent stochastic trends. The horizontal axis indicates the subsample sizes from 50 to 100, the vertical axis indicates the respective upper and lower bounds of the confidence intervals as well as the test statistics over the whole sample. These graphs show that for the 5 smallest eigenvalues the test statistics and the confidence intervals are of the same order and in most cases the test statistics lie within the corresponding confidence intervals, while for the other 11 larger eigenvalues the test statistics and the subsample interval bounds are not of the same order, and the test statistics are about T/b times larger (in absolute value) than the corresponding bounds. These graphs provide a clear hint why the power of the test is so high.

In case of the presence of moderate autocorrelation¹⁵ in the stationary com-

¹⁵It is to note that if the autocorrelation in u_t are getting close to units, subsampling will not have good results with limited number of observations. This is due to the same reason why Dick-fuller test can not differ the unit root process from a AR process with a

ponents (u_t is autocorrelated.) this testing procedure works still quite well. The next panel of graphs are simulation results with following parameters: $T = 200, n = 16, g = 5, b \in (50, 100), \rho_1 = 0.4, \rho_2 = 0.3, \rho_3 = -0.2, \rho_4 = 0.1$

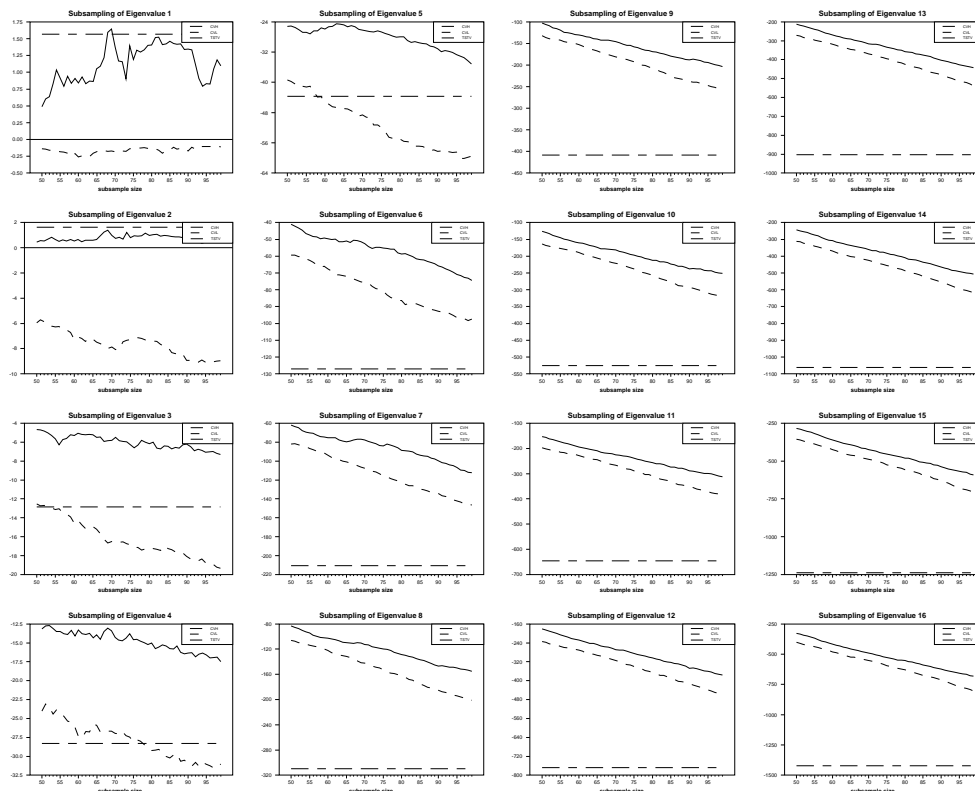


Figure 2: *Subsampling Confidence Interval with Autocorrelation*

In the following we compare our testing procedure with Johansen's ML approach. The reason for taking this procedure as the benchmark for the comparison with the subsampling procedure is that Johansen procedure seems to be the most popular one in applied cointegration research¹⁶. Further, the H_0 under test are the same - the number of independent stochastic trends.

autocorrelation coefficient close to unit if the number of observations are limited.

¹⁶There are some other methods: Stock and Watson (1988), Phillips (1991), Bierens (1997), and Bierens (1997) which can be used to test for the number of cointegration relations. See Haug (1996) for a study of the comparison of some of these methods.

No.	Method	n	g_0	T	b	$H_0 : g_0 - 1$	$H_0 : g_0$	$H_0 : g_0 + 1$	$H_0 : g_0 + 2$
1	SS	15	5	200	90	0.81	0.90	0.0	0.0
1	JH	15	5	200	-	0.95	0.87	0.02	0.0
2	SS	20	5	200	90	0.75	0.85	0.00	0.0
2	JH	20	5	200	-	0.97	0.85	0.05	0.03
3	SS	30	5	200	90	0.85	0.87	0.00	0.0
3	JH	30	5	200	-	0.98	0.86	0.22	0.10
4	SS	40	5	200	90	0.86	0.85	0.00	0.0
4	JH	40	5	200	-	0.98	0.95	0.66	0.16
5	SS	50	5	200	90	0.81	0.83	0.00	0.0
5	JH	50	5	200	-	0.98	0.97	0.88	0.68
6	SS	60	5	200	90	0.87	0.68	0.00	0.0
6	JH	60	5	200	-	0.98	0.98	0.98	0.97

Table 3: *Relative frequency of acceptance of H_0 at the nominal level of 90%. n is the dimension of the system. g_0 is the number of the independent stochastic trends. T is the total sample size. b is the subsample size. H_0 under test are $g = g_0 - 1, g = g_0, g = g_0 + 1$ and $g = g_0 + 2$ respectively. $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0, \rho_{e1} = \rho_{e2} = \rho_{e3} = \rho_{e4} = 0.$, the lag in the VAR of the ECM of Johansen procedure is 1. SS indicates subsampling method. JH indicates Johansen procedure. Same number in the first column means the test are carried out using same set of data.*

Obviously, the power of Johansen procedure in rejection of false stochastic trends is low. This tendency get worse when the dimension of the system becomes large. Table 4 presents the simulation results of the comparison between Johansen procedure and the subsampling procedure at the presence of autocorrelation in the stationary components.

No.	Method	n	g	T	b	$H_0 : g_0 - 1$	$H_0 : g_0$	$H_0 : g_0 + 1$	$H_0 : g_0 + 2$
1	SS	15	5	200	110	0.81	0.88	0.00	0.0
1	JH	15	5	200	-	0.99	0.95	0.93	0.87
2	SS	20	5	200	110	0.78	0.80	0.00	0.00
2	JH	20	5	200	-	0.99	1.00	0.97	0.90
3	SS	30	5	200	110	0.84	0.61	0.00	0.00
3	JH	30	5	200	-	0.92	0.87	0.77	0.56
4	SS	40	5	300	110	0.82	0.63	0.00	0.00
4	JH	40	5	300	-	0.99	0.99	0.99	0.99
5	SS	50	5	300	150	0.86	0.74	0.00	0.00
5	JH	40	5	300	-	0.97	0.91	0.81	0.73
6	SS	50	5	300	150	0.89	0.63	0.00	0.00
6	JH	60	5	300	-	0.99	0.99	0.99	0.99

Table 4: *Relative Frequency of Acceptance of H_0 at the nominal level 90%. n is sample size. g_0 is number of independent stochastic trends of the DGP. H_0 under test are $g_0 - 1, g_0, g_0 + 1$ and $g_0 + 2$ respectively. T is total sample size. b is subsample size. $\rho_1 = 0.4, \rho_2 = 0.3, \rho_3 = -0.2, \rho_4 = 0.1$, The lags in the VAR of the ECM of Johansen procedure is 5.*

This Monte Carlo analysis show to some extent that in the cases of our simulation with limited number of observations Johansen procedure has extremely low power in the sense that it accepts falsely too many independent stochastic trends. The situation gets worse when the autocorrelation is present in the stationary component of the DGP. In these cases the subsampling procedure performs better than the Johansen procedure. Of course this Monte Carlo analysis does not provide sufficient evidence that the subsampling approach always works better than Johansen method. As can be seen in Phillips (1991) the null distribution has strong impact on the performance of the test. One may suspect that the reason for the poor power of the Johansen procedure may be the null distribution of the simulated data have not been formulated in a proper error correction form. To clarify this question, we apply next the subsampling test procedure and Johansen procedure to data that are generated from ECMs:

$$y_t = \beta\alpha'y_{t-1} + \pi_1\Delta y_{t-1} + \epsilon_t \quad (5.24)$$

where y_t is a $n \times 1$ vector of time series, α and β are $n \times h$ matrix, $h < n$ and ϵ is a $n \times 1$ gaussian white noise process. Following are some simulation results.

No.	Method	n	g	T	b	$H_0 : g_0 - 1$	$H_0 : g_0$	$H_0 : g_0 + 1$	$H_0 : g_0 + 2$
1	SS	15	5	200	100	0.67	0.72	0.00	0.0
1	JH	15	5	200	-	0.98	0.84	0.28	0.0
2	SS	20	5	200	100	0.59	0.70	0.00	0.0
2	JH	20	5	200	-	0.97	0.91	0.40	0.03
3	SS	30	5	200	100	0.79	0.56	0.00	0.0
3	JH	30	5	200	-	0.98	0.96	0.71	0.27
4	SS	40	5	200	100	0.78	0.59	0.00	0.0
4	JH	40	5	200	-	0.99	0.97	0.93	0.85
5	SS	50	5	300	150	0.78	0.72	0.00	0.0
5	JH	50	5	-	-	0.98	0.92	0.57	0.06
6	SS	60	5	300	150	0.74	0.61	0.00	0.0
6	JH	60	5	300	-	1.0	0.97	0.78	0.38

Table 5: *Relative frequency of acceptance of H_0 at the nominal level of 90%. n is the dimension of the system. g_0 is the number of independent stochastic trends of the DGP. H_0 under test are $g = g_0 - 1, g = g_0, g = g_0 + 1$ and $g = g_0 + 2$ respectively. T is the total sample size. b is the used subsample size. The lags in the VAR of the ECM of Johansen procedure are 2.*

Table 5 shows that even when data are generated by ECMs, Johansen Procedure tends to have low power in large systems. while the power of the subsampling procedure keeps to be high, it has an overall downwards size distortion in these set of simulations.

The reason for this size distortion of the subsampling procedure and the over-acceptance of Johansen procedure is that both procedures are asymptotically valid tests. They will behave well when the sample size is sufficiently large or equivalently when the eigenvalues whose limits are zero are close to zero. When the dimension of a system under investigation become large and the sample size is not large enough, the calculated eigenvalues tend to be large than zero in case of Johansen procedure and less than zero in case of subsampling. Thus in case of Johansen procedure the corresponding trace statistic become smaller, which leads to overacceptance of H_0 in Johansen procedure, while in case of subsampling the smaller statistic leads to overrejection of H_0 at the chosen subsample size¹⁷.

6 An Application Example

After we investigate the performance of subsampling testing procedure, we turn to empirically testing of PPP. Purchasing power parity is a benchmark

¹⁷As mentioned before data driven optimal sample size may help to improve the size problem in subsampling.

model for nominal exchange rate determination. Over last decades a vast amount of empirical works have been conducted to assess the evidence of PPP in the real world. While most literature in 80s found evidence against PPP: the real exchange rate were found to follow a random walk, see for example Frankel (1981), and Enders (1988); the works in 90s deliver some results favor PPP in a long run cointegration relation, see for example Johansen and Juselius (1992), Kim (1990) and Abuaf and Jorion (1990). Resent panel studies do not provide a clear picture of PPP. For example Choi (2001), Oh (1996) found evidence favor PPP, while Fleissig and Strauss (2000) did not find evidence favor PPP.

The economical reasoning behind PPP is the goods arbitrage, i.e. the domestic price for products should equal to the foreign price for the same foreign products converted by the exchange rate:

$$P_d = S_{df} P_f, \quad (6.25)$$

where P_d is the domestic price, P_f is the foreign price, and S_{df} is the the price of foreign currency - the exchange rate.

Taking the logarithm we get :

$$p_d = s_{df} + p_f, \quad (6.26)$$

where the small case letters denote the corresponding logs. Because p_d , p_f and s_{df} are I(1) variables the exact PPP relation (6.26) is then tested in the form that the real exchange rate s_{df}^r is a stationary I(0) variable:

$$s_{df}^r := s_{df} - p_d + p_f. \quad (6.27)$$

This implies at same time that the variable p_d , p_f and s_{df} are cointegrated with a particular cointegration vector $(1, -1, 1)'$. A large number of empirical studies were conducted in 90s to explore this specific cointegration relation among nominal exchange rates and price indices between different countries. Also in the panel studies, this cointegration relation is the object under test. Bilateral PPP implies many cointegration relations in a multi-country system.

In the following we are going to characterize PPP for a system of N countries. In the case of three countries, we may haver pair wise PPP:

$$-p_0 + s_{01} + p_1 \sim I(0), \quad (6.28)$$

$$-p_0 + s_{02} + p_2 \sim I(0), \quad (6.29)$$

and

$$-p_1 + s_{12} + p_2 \sim I(0). \quad (6.30)$$

In fact (6.30) is the difference between (6.29) and (6.28). Therefore PPP implies two linear independent particular cointegration relations among three countries. This simple example can be generalized to the case of N countries. In fact bilateral PPP relations with respect to one country implies bilateral ppp relations among other countries. PPP implies $N - 1$ independent cointegration relations among a system of N countries with $2N - 1$ variables. Choose arbitrarily a country as the base country, we may order the relevant variables of the system in the following way:

$$y_t = \begin{pmatrix} p_0 \\ s_{01} \\ p_1 \\ \vdots \\ s_{0(N-1)} \\ p_{N-1} \end{pmatrix}. \quad (6.31)$$

Then PPP implies that this $2N - 1$ system is a cointegrated system with $N - 1$ cointegration relations like follows:

$$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 & & \dots & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & & \vdots \\ \vdots & \vdots & 0 & 0 & 0 & \ddots & \ddots & 0 \\ -1 & 0 & \dots & & & & 0 & 1 & 1 \end{pmatrix}. \quad (6.32)$$

Taking advantage of these particular cointegration relations, many empirical works were conducted in panel setting to increase the power of test.

In this paper we conduct a direct test of these $N - 1$ cointegration relations in the $2N - 1$ dimensional system. Here we investigate the relation among exchange rates and price levels among the most important 10 currencies in the world economy: US Dollars, Canadian Dollars, Euro, Japan Yen, Swiss Francs, British Pounds, Korea Wan, Australian Dollars, New Zealand Dollars, Mexican Pesos¹⁸. We look at a system of 9 exchange rates and 10 price indices. The data are daily from 2000:01:01 to 2002:10:02. The main summary statistics of the data are in the following table.

Series	Obs	Mean	Std Error	Minimum	Maximum
CAD_USD	717	1.50130098	0.07950400	1.29240000	1.61320000
DEUR_USD	717	1.03754449	0.10297501	0.79386000	1.20871000
JPY_USD	717	117.61344491	7.68910553	102.65000000	134.79000000
GBP_USD	717	0.65868662	0.03765339	0.56031000	0.72563000

¹⁸There are still two important currencies: the Hongkong Dollars and the Chinese Yuan. Because they were institutionally under pegging system, we do not consider these two currencies.

CHF_USD	717	1.57014533	0.15506100	1.23780000	1.82140000
AUD_USD	717	1.76061144	0.17266568	1.32810000	2.06980000
NZD_USD	717	2.11555411	0.27333018	1.52620000	2.54700000
KRW_USD	717	1214.26457462	68.86227937	1105.20000000	1368.30000000
MXN_USD	717	9.81171116	0.65366391	9.00230000	11.40970000
PCANADA	717	1.11234310	0.02287517	1.07000000	1.16000000
PEURO	717	1.12616457	0.02159016	1.09000000	1.16000000
PJAPAN	717	1.00903766	0.00834286	0.99000000	1.02000000
PUK	717	1.15896792	0.01697173	1.12000000	1.19000000
PSWISS	717	1.04523013	0.00857818	1.03000000	1.06000000
PAUS	717	1.13938633	0.03681317	1.07000000	1.19000000
PNZ	717	1.10020921	0.02380873	1.06000000	1.14000000
PKOREA	717	1.25447699	0.03443740	1.20000000	1.31000000
PMEXIKO	717	2.52722455	0.11249691	2.32000000	2.72000000
PUS	717	1.15408647	0.02211974	1.11000000	1.19000000

Source : Federal Reserve Bank of St. Louis
<http://research.stlouisfed.org/fred2/categories/15> \\
OECD: main economic indicator

All 19 series are graphed in the following graph panel. All series show strong stochastic trends. Except Japan all price indices have upward trend. Some exchange rates such as EURO/USD, BRP/USD, CHF/USD, AUD/USD and NZD/USD show well pronounced similar trend movement, which may suggest some cointegration relations among them.

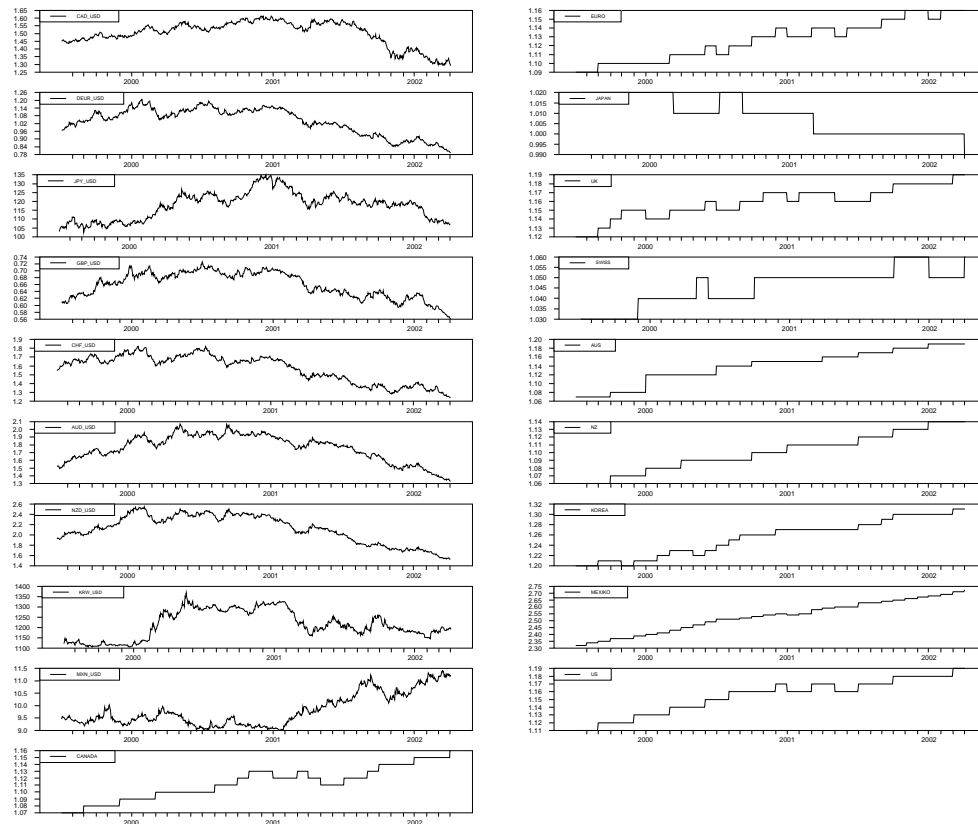


Figure 3: The fundamental data of the model.

Our testing result, contrary to the optical impression, show that there is no cointegration relations among these 19 variables. In Figure 4 we can see clearly the even the smallest eigenvalue(in absolute value the largest) has the same order of the corresponding confidence bounds (see the last graph in the fourth column): it lies in fact within the subsampling confidence interval. This mean that there is no cointegration relations within 19 variables. This result implies that we can not find empirical evidence for PPP hypothesis with daily data within a short period of 3 years. This empirical result confirms the findings in the literature that there is no evidence for PPP for short run period.

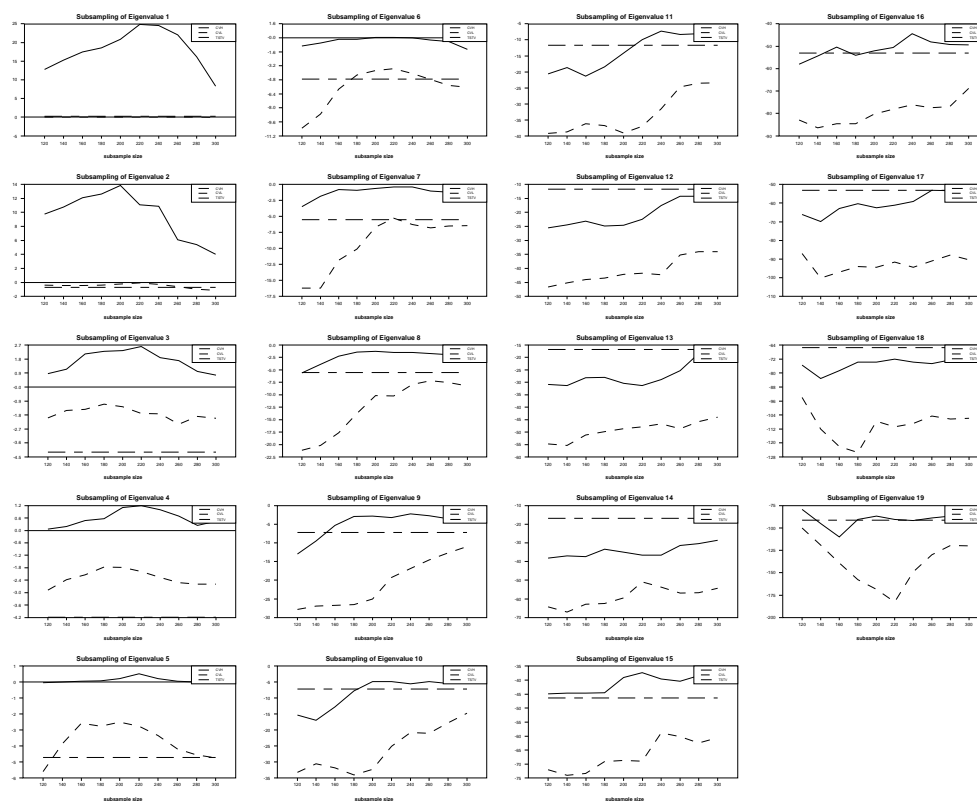


Figure 4: Subsampling confidence interval for block size: $b \in (100, 300)$.

Now we turn to the long run PPP relation. We investigate in fact the same system in the post-Bretton Woods period from 1973 to 2002. Due to data availability Korea and Mexico have to be excluded from the system. The data are monthly from 1973:1 to 2001:12. The main summary statistics of the data are in the following table.

Series	Obs	Mean	Std Error	Minimum	Maximum
CAD_USD	348	1.25128994	0.15740167	0.96230000	1.59220000
JPY_USD	348	181.40913793	68.00074320	83.69000000	305.67000000
GBP_USD	348	0.58802093	0.09610155	0.38816862	0.91482938
CHF_USD	348	1.83921638	0.52598882	1.13840000	3.72930000
DEM_USD	348	2.05346236	0.43630573	1.38120000	3.30250000
AUD_USD	348	1.21006869	0.31927103	0.67317402	1.99362041
NZD_USD	348	1.50079774	0.44567899	0.67276642	2.50626566
CAN445241K	348	0.74470489	0.26868023	0.25810000	1.12750000
GEM125241K	348	0.80161207	0.18652050	0.45510000	1.10210000
JAP465241K	348	0.84753764	0.17127051	0.37260000	1.03340000
SWZ685241K	348	0.78951437	0.18556799	0.44870000	1.05450000
USA425241K	348	0.74622989	0.26400929	0.28000000	1.17000000

GRB265241K	348	0.69524195	0.31238563	0.15190000	1.17140000
NZL595241K	348	0.65978621	0.33980043	0.12400000	1.11270000
AUS545241K	348	0.68831379	0.30557015	0.17560000	1.15970000

Source : Federal Reserve Bank of St. Louis
<http://research.stlouisfed.org/fred2/categories/15\\>
OECD: main economic indicator

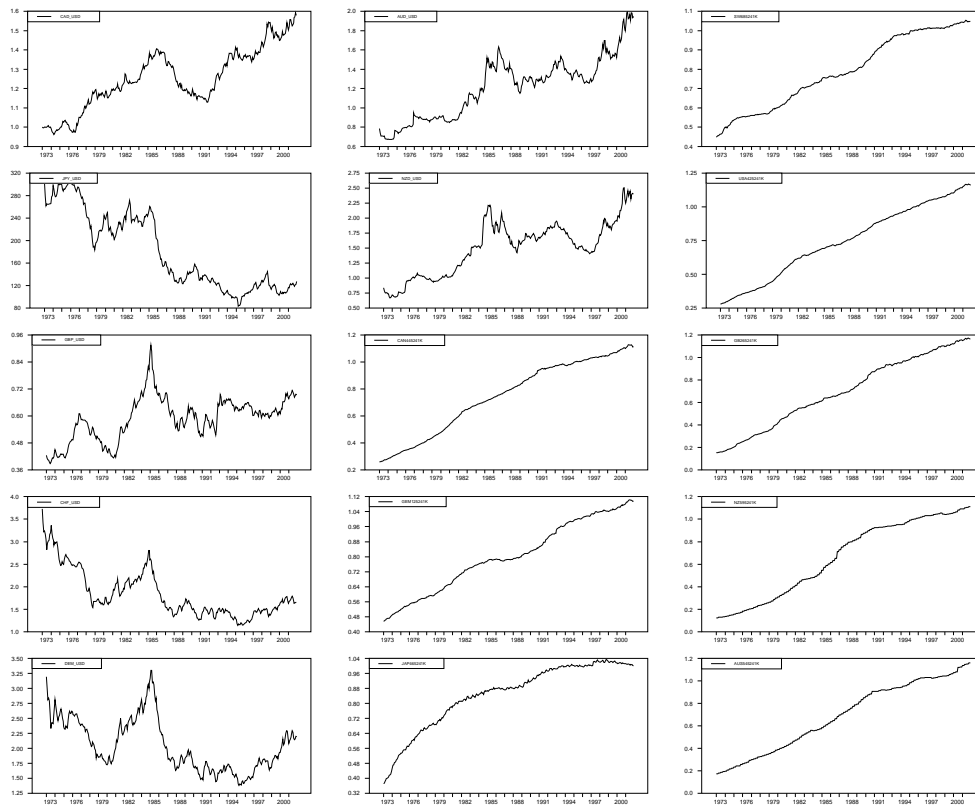


Figure 5: The fundamental data of the model.

On the graphs above there are well pronounced comovement among many series. We would expect some cointegration relation among them. Contrary to the optical impression, the test shows that there is no cointegration relations among these 15 variables. The subsampling test results are depicted in the following Figure 6.

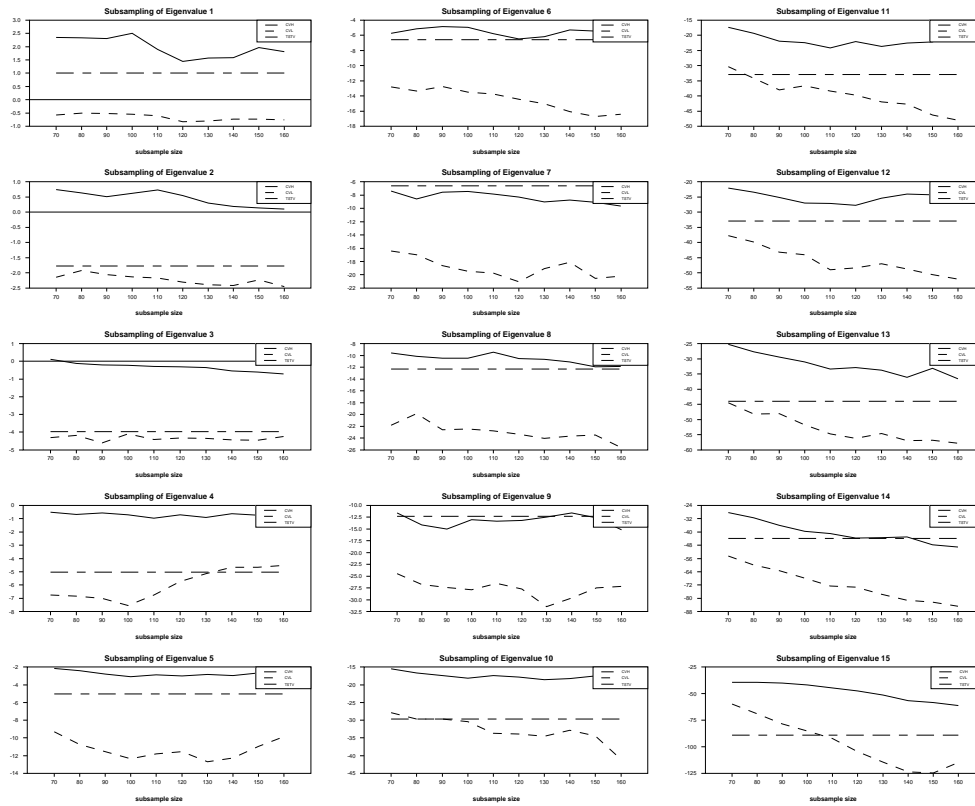


Figure 6: Subsampling confidence interval for block size: $b \in (70, 160)$.

In the graphs above we can not see any hint of diverging property of the eigenvalues. Even the smallest eigenvalue has the same order as the the subsample bounds. Therefore we conclude that the system does not contain any cointegration relations. This means that PPP hypothesis is rejected in this set of data.

7 Concluding Remarks

In this paper we develop a procedure to explore the cointegration relations among large numbers of variables and thus contribute to the econometric literature on multivariate cointegration analysis in large systems. The non-parametric subsampling test procedure is shown to be asymptotically most powerful and its performance is quit satisfactory with moderate sample size for small number of stochastic trends.

However, subsampling test is only a asymptotically valid test. The performance of the test in limited sample cases may depend on the parameter of

the testing procedure such as the block size and the parameters of the DGP. Instead of searching for a optimal subsample size, we apply a range of reasonable subsample sizes to conduct our conclusion based on the qualitative difference between the divergence of the test statistics and the convergence of the test statistics.

Simulation studies show that the subsampling procedure perform very well for the cases of large systems with few independent stochastic trends with moderate requirement on data.

Further researches is planed to explore the limits of the application of subsampling to testing of cointegration ranks in different extreme situations:

- when the number of independent stochastic trend is approaching to the dimension of the model,
- when the autocorrelation of the stationary part u_t is approaching to unit and
- the number of total observations is approaching the dimension of the DGP.

8 Appendix

Proof of Lemma 3.3.

Using the continuous mapping theorem we have

$$\left(\frac{G_{iT}}{T^\alpha}\right)^{-1} = \frac{G_{iT}^{-1}}{T^{-\alpha}} \Rightarrow G_i^{-1}.$$

(i) is proved.

For (ii),

$$\frac{G_{iT}}{T^{\alpha_i}} \frac{G_{jT}}{T^{\alpha_j}} = \frac{G_{iT} \cdot G_{jT}}{T^{\alpha_i + \alpha_j}} \xrightarrow{D} G_i \cdot G_j.$$

The proof for (iii) is similar.

□

Lemma 8.1

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Proof of Lemma 8.1 is done easily by matrix multiplication of the expression above and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

□

Proof of Lemma 3.4

First, we calculate

$$\begin{pmatrix} G_{1,T} & G_{2,T} \\ G_{3,T} & G_{4,T} \end{pmatrix}^{-1} =: \begin{pmatrix} G_{1,T}^* & G_{2,T}^* \\ G_{3,T}^* & G_{4,T}^* \end{pmatrix}.$$

Using Lemma 8.1, the first block matrix is equal to

$$\begin{aligned} (G_{1,T}^*)^{-1} &= T \frac{G_{1,T}}{T} - T \frac{G_{2,T}}{T} T^{-2} \left(\frac{G_{4,T}}{T^2}\right)^{-1} T \frac{G_{3,T}}{T} \\ &= T \underbrace{\frac{G_{1,T}}{T}}_{\xrightarrow{P} g_1} - \underbrace{\frac{G_{2,T}}{T} \left(\frac{G_{4,T}}{T^2}\right)^{-1} \frac{G_{3,T}}{T}}_{\xrightarrow{D} G_2 G_4^{-1} G_3}. \end{aligned}$$

The convergence of the second term in the last equation is guaranteed by the continuous mapping theorem, see Hamilton (1994) P.482. Then,

$$\frac{G_{1,T}^*}{T^{-1}} \xrightarrow{P} \lim_{T \rightarrow \infty} \frac{(g_1 T - G_2 G_4^{-1} G_3)^{-1}}{T^{-1}} = g_1^{-1}.$$

Similarly, we have

$$\begin{aligned}\frac{G_{2,T}^*}{T^{-2}} &\xrightarrow{D} -g_1^{-1}G_2G_4^{-1} \\ \frac{G_{3,T}^*}{T^{-2}} &\xrightarrow{D} -G_4^{-1}G_3g_1^{-1} \\ \frac{G_{4,T}^*}{T^{-2}} &\xrightarrow{D} G_4^{-1}.\end{aligned}$$

Now we calculate the limit of $G_{7,T}, G_{8,T}$. Due to the continuous mapping theorem we are allowed to have the following "calculation"

$$\begin{aligned}\begin{pmatrix} G_{1,T} & G_{2,T} \\ G_{3,T} & G_{4,T} \end{pmatrix}^{-1} \begin{pmatrix} G_{5,T} \\ G_{6,T} \end{pmatrix} &\sim \begin{pmatrix} g_1^{-1}T^{-1} & -g_1^{-1}G_2G_4^{-1}T^{-2} \\ -G_4^{-1}G_3g_1^{-1}T^{-2} & G_4^{-1}T^{-2} \end{pmatrix} \begin{pmatrix} g_5T \\ G_6T \end{pmatrix} \\ &\sim \begin{pmatrix} g_1^{-1}g_5 \\ (-g_5G_4^{-1}G_2g_1 + G_6G_4)T^{-1} \end{pmatrix}\end{aligned}$$

□

Lemma 8.2 *Let a, b, c, d be $h \times h, h \times g, g \times h, g \times g$ given matrices. Let $B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ where $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are $h \times h, h \times g, g \times h, g \times g$ given matrices. Then, the eigenvalues of the matrix*

$$B \begin{pmatrix} Ta & Tb \\ c & d \end{pmatrix} \tag{8.33}$$

are separated into two groups: one has the order T and the other one has the order 1.

Note Lemma 8.2 will be applied later for $T \rightarrow \infty$.

Proof of the Lemma 8.2 The idea of the proof is that the eigenvalues of the matrix X are invariant under the transformation $\Psi^{-1}X\Psi$.

Search Ψ_T of the form

$$\Psi_T = \begin{pmatrix} I & e_{12,T} \\ e_{21,T} & I \end{pmatrix}$$

where $e_{12,T}$ and $e_{21,T}$ are $h \times g$ and $g \times h$ matrices respectively.

From straightforward calculation, we obtain

$$\Psi_T^{-1} B \begin{pmatrix} Ta & Tb \\ c & d \end{pmatrix} \Psi_T = \begin{pmatrix} \tilde{A}_T & \tilde{B}_T \\ \tilde{C}_T & \tilde{D}_T \end{pmatrix}$$

where

$$\tilde{A}_T = (I - e_{12,T}e_{21,T})^{-1} \left[\begin{array}{l} T(\beta_{11} - e_{12,T}\beta_{21})(a + be_{21,T}) \\ + (\beta_{12} + e_{12,T}\beta_{22})(c + de_{21,T}) \end{array} \right], \quad (8.34)$$

$$\tilde{B}_T = (I - e_{12,T}e_{21,T})^{-1} \left[\begin{array}{l} T(\beta_{11} - e_{12,T}\beta_{21})(ae_{12,T} + b) \\ + (\beta_{12} + e_{12,T}\beta_{22})(ce_{12,T} + d) \end{array} \right], \quad (8.35)$$

$$\tilde{C}_T = (I - e_{21,T}e_{12,T})^{-1} \left[\begin{array}{l} T(-e_{21,T}\beta_{11} + \beta_{21})(a + be_{21,T}) \\ + (-e_{21,T}\beta_{12} + \beta_{22})(c + de_{21,T}) \end{array} \right], \quad (8.36)$$

$$\tilde{D}_T = (I - e_{21,T}e_{12,T})^{-1} \left[\begin{array}{l} T(-e_{21,T}\beta_{11} + \beta_{21})(ae_{12,T} + b) \\ + (-e_{21,T}\beta_{12} + \beta_{22})(ce_{12,T} + d) \end{array} \right]. \quad (8.37)$$

We choose $e_{12,T}$ and $e_{21,T}$ such that $\tilde{B}_T \sim 0$ and $\tilde{D}_T \sim 0$.

From the constraint $\tilde{B}_T \sim 0$, we choose $e_{12,T}$ with the expression

$$e_{12,T} = -a^{-1}b + \frac{\Delta_{1,T}}{T} + O(T^{-2}) \quad (8.38)$$

where

$$\Delta_{1,T} = a^{-1}(\beta_{11} - \hat{e}_{12}\beta_{21})^{-1}(\beta_{12} + \hat{e}_{12}\beta_{22})(c\hat{e}_{12} + d)$$

with $\hat{e}_{12} = a^{-1}b$. The expression (8.38) provides an approximation

$$ae_{12,T} + b = O(T^{-1}). \quad (8.39)$$

Similarly, under the setting $\tilde{C}_T \sim 0$, we choose $e_{21,T}$ with the expression

$$e_{21,T} = \beta_{21}\beta_{11}^{-1} + \frac{\Delta_{2,T}}{T} + O(T^{-2}) \quad (8.40)$$

where

$$\Delta_{2,T} = (-\hat{e}_{21}\beta_{12} + \beta_{22})(c + d\hat{e}_{21})(a + b\hat{e}_{21})^{-1}\beta_{11}^{-1}$$

with $\hat{e}_{21} = \beta_{21}\beta_{11}^{-1}$. This expression provides an approximation

$$-e_{21,T}\beta_{11} + \beta_{21} = O(T^{-1}). \quad (8.41)$$

With the chosen $e_{12,T}$ and $e_{21,T}$, the leading term in (8.34) is the first term. Thus

$$\lim_{T \rightarrow \infty} \frac{\tilde{A}_T}{T} = (\beta_{11} - a^{-1}b\beta_{21})(a + b\beta_{21}\beta_{11}) =: A, \quad (8.42)$$

so \tilde{A}_T has order T . Let $\lambda_{1,T}, \dots, \lambda_{h,T}$ be the eigenvalues of \tilde{A}_T and $\lambda_1, \dots, \lambda_h$ be the eigenvalues of A in descending order in absolute value. Then due to the continuity of the eigenvalues

$$\lim_{T \rightarrow \infty} \frac{\lambda_{i,T}}{T} = \lambda_i, \quad \text{for } i = 1, \dots, h.$$

Now, we calculate the eigenvalues of \tilde{D}_T . The most tricky part of this prove is that, by applying the approximations (8.39) and (8.41), the first term on the RHS in Eq. (8.37) has order $\frac{1}{T}$ instead of T , therefore the leading term of \tilde{D}_T in Eq. (8.37) is the second term. Then,

$$\lim_{T \rightarrow \infty} \tilde{D}_T = (-\beta_{21}\beta_{11}^{-1}\beta_{12} + \beta_{22})(ca^{-1}b + d) =: D. \quad (8.43)$$

The eigenvalues of \tilde{D}_T converges to the eigenvalues of D .

□

Proof of Theorem 3.5

Theorem 3.5 is the stochastic version of Lemma 8.2. In order to use Lemma 8.2, we correspond $\begin{pmatrix} Ta & Tb \\ c & d \end{pmatrix}$ Eq. in (8.33) to $\begin{pmatrix} TG_{7,T} \\ TG_{8,T} \end{pmatrix}$ in Eq. (3.9). Since the correspondence of a, b, c, d are random variables and dependent on T , we add the subindex T to a, b, c, d and define

$$\begin{aligned} a_T &= G_{7,T,1} & b_T &= G_{7,T,2} \\ c_T &= TG_{8,T,1} & d_T &= TG_{8,T,2}. \end{aligned}$$

From Lemma 3.4 we have the convergence

$$\begin{aligned} (a_T \ b_T) &\xrightarrow{P} g_7 =: (g_{7,1} \ g_{7,2}) \\ (c_T \ d_T) &\xrightarrow{D} G_8 =: (G_{8,1} \ G_{8,2}). \end{aligned}$$

These convergence has a strong version of convergence almost surely, see Föllmer (1981), that means the convergence

$$\begin{aligned} (a_T \ b_T) &\longrightarrow g_7 =: (g_{7,1} \ g_{7,2}) \\ (c_T \ d_T) &\longrightarrow G_8 =: (G_{8,1} \ G_{8,2}) \end{aligned}$$

holds for almost all paths.

Applying Lemma 8.2 for all the path satisfying the convergence above, and define

$$A := \lim_{T \rightarrow \infty} \frac{\tilde{A}_T}{T} = (\beta_{11} - g_{7,1}^{-1}g_{7,2}\beta_{21})(g_{7,1} + g_{7,2}\beta_{21}\beta_{11}) \quad (8.44)$$

$$D := \lim_{T \rightarrow \infty} \tilde{D}_T = (-\beta_{21}\beta_{11}^{-1}\beta_{12} + \beta_{22})(G_{8,1}g_{7,1}^{-1}g_{7,2} + G_{8,2}). \quad (8.45)$$

Then, A and D are well-defined for almost all paths (with probability one). Then, the eigenvalues of $\frac{\tilde{A}_T}{T}$ and \tilde{D}_T converge to the eigenvalues of A and D with probability one. The statements (3.12) and (3.13) in Theorem 3.5 are proved.

□

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