First draft June 1, 1999; Second draft November 12, 2004

Model Uncertainty, Complexity and Rank in Finance

Abstract

There are three crucial mathematical system concepts in Finance, which are either being confused or misapplied - uncertainty, complexity and rank. First, the concept of epistemic uncertainty is sufficient for modeling and the concept of probability is unnecessary. This is illustrated by "Galton's Error," and the under-repesentation of systematic risk by American mutual funds. These funds use simple unidirectional projection ("regression") to compute Sharpe's beta for fund selection. There are at least five equivalent ways of representing the measured model uncertainty and a new and an improved risk categorization for mutual funds is presented. Second, the concept of (linear) system complexity is usually dealt with by presuming a model rank, as the Cowles Foundation erroneously prescribed in the early 1950s, and superimposing that model rank on the data, when a model is estimated. But the model rank does not have to be presumed: it can be identified from the data and all corresponding (Grassmanian) coefficients can be computed by CLS Projections. This is illustrated by the identification of the model rank of simple financial risk systems in six Asian countries, in particular in Taiwan. Third, often it is thought that Markowitz' portfolio optimization and exact and complete cash flow accounting are incompatible because of the non-existence, or empirical instability of the information matrix. The problem is caused by the rank constraints imposed by the portfolio accounting identities. But these rank constraints also provide the solution, since they form exact selectors of the portfolio allocations, which are found by simple tensor algebra. This will be illustrated by the optimization of an Asian multi-currency stock investment portfolio.

Acknowledgement 1 *The ideas for this paper were first outlined in an invited lecture on December 21, 1990, at the University of Zürich, and were presented in the current form in a seminar on December 2, 1997 at the University of Adelaide and have been included in several of my articles, but are here brought together to demonstrate that these scientific ideas belong together: they are attempts to extinguish unfounded prejudices in Finance. With thanks to my research students Au Yin Fung, Lee Lin Kew, Ong Sze Wei, Ho Su-Ping, Jeyanthi Karuppiah and Lim Cheer Hwi for their data gathering and computations, while I lectured at the Nanyang Technological University in Singapore in 1995-99.*

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1 INTRODUCTION

In the past half century, financial theory and practice have developed rapidly. In the 1960s the Capital Asset Pricing Model (CAPM) was introduced, together with professional mutual fund management. In the 1970s derivatives pricing theory was introduced, and options trading started its meteoric rise. In the 1980s Asset Pricing Theory (APT) and more complex risk measurement (like *Riskmetrics*) and valuation methods, like Value-at-Risk (VaR), were introduced. And, in the 1990s, after several bank lending crises, the even more exotic measurement and analysis of credit risk, modeled financial market risk. For a survey of these developments any good finance textbook will suffice.

To sustain its rapid growth, the financial discipline liberally borrowed methodological concepts from other disciplines, *e.g.*, from mathematics, statistics, econometrics, mathematical system analysis, and from signal processing. This phenomenon has even accelerated under the impetus of financial engineering and scientific measurement of all types of hedging, portfolio and credit risks. But not all such concepts are properly transferred and used. In fact, there are three interrelated modeling concepts in Finance, which are either confused or misapplied - uncertainty, complexity, and rank. All three concepts are mathematical system concepts. Their confusion and misapplication has caused havoc in both the financial and economic literature of the past half century. This paper attempts to correct and redirect those research effort.

In this Introduction, we first define these fundamental concepts in simple mathematical terms. These concepts wee developed in Los (1989a & b; 1992) Next, each of these concepts will be discussed in greater detail in the three following sections and illustrated by some real world examples. For simplicity, the discussion of these concepts is presented in the static context of stationary processes. However, it can easily be extended to non-stationary data, with time-varying first and second moments, which can be filtered by the usual Kalman filter. Of course higher-order data non-stationarity must be analyzed and modeled differently and is the subject of Los (2003) and Jamdee, Los and Yalamova (2004).

First, we define mathematically what is meant in this paper by data input and by linear model. The *data input* is the $n \times n$ symmetric data covariance matrix $\mathbf{\Sigma}$ of averaged products of a $T \times n$ matrix $\mathbf{x}$ of deviations of n time series variables from their respective means, which are observed at T regularly spaced time moments. The innocuous elimination of averages simplifies the analysis. The diagonal elements of this covariance matrix, σ_{ii}, are variances, while its off-diagonal elements, σ_{ij} for $i \neq j$, are covariances. There are $\frac{n(n-1)}{2}$ such independent bivariate covariances. For example, for the bivariate case we have:

$$\mathbf{\Sigma} = \frac{\mathbf{x}'\mathbf{x}}{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{1}$$

For *exact data* the covariance matrix $\mathbf{\Sigma}$ is singular, because of the exact linear dependencies among the variables. But for *empirical inexact data* the symmetric data covariance matrix is generically positive definite, and, consequently, nonsingular.

By a *linear model* is meant a model linear in its coordinates, or coefficients, since the model's variables can be unique (nonlinear) transformations of the original measurements. The linear model is generically defined by the expression

$$\mathbf{A}\widehat{\mathbf{x}}' = \mathbf{0} \tag{2}$$

with $\widehat{\mathbf{x}} = \mathbf{x} - \widetilde{\mathbf{x}}$ such that $\widehat{\mathbf{x}} \perp \widetilde{\mathbf{x}}$, since what is known of the data, *i.e.*, explained by the linear model (= linear dependency), $\widehat{\mathbf{x}}$, is orthogonal to what is unknown, $\widetilde{\mathbf{x}}$. Consequently, also $\mathbf{A}\widehat{\mathbf{\Sigma}} = \mathbf{0}$, where the systematic covariance matrix (= matrix of linear dependencies) $\widehat{\mathbf{\Sigma}} = \mathbf{\Sigma} - \widetilde{\mathbf{\Sigma}}$, with $\left|\widehat{\mathbf{\Sigma}}\right| = 0$ and $\widetilde{\mathbf{\Sigma}}$ is the unsystematic covariance matrix (= noise matrix), with $\widetilde{\mathbf{\Sigma}} \geq \mathbf{0}$, positive semi-definite, *i.e.*, not necessarily invertible. Technically, $\mathbf{A}$ is the $q \times n$ matrix containing the computed dual Grassmanian coordinates, conventionally known as the "model coefficients." The system invariant or corank q (= number of independent linear relations in the exact model) has to be determined from the inexact data for $n > 2$, where n = number of variables, since, generically, $1 \leq q < n$.

The three crucial concepts of modeling uncertainty, complexity and rank in finance modeling can now also be defined as follows:

1. *Modeling uncertainty* exists, when the determinant of the data covariance matrix is positive, $|\mathbf{\Sigma}| > 0$. This modeling uncertainty, or incompleteness, can be measured by the Noise/Data ratio:

$$\frac{N}{D} = \frac{|\mathbf{\Sigma}|}{\prod_i^n \sigma_{ii}} \geq 0 \tag{3}$$

For example, for the bivariate case, $n = 2$, the percentage noise in the data is $\frac{N}{D} = \frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{\sigma_{11}\sigma_{222}} = 1 - \rho_{12}^2$, where the coefficient of determination $\rho_{12}^2 = \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}$ and the noise/signal ratio $\frac{N}{S} = \frac{1-\rho_{12}^2}{\rho_{12}^2}$. There exists no modeling uncertainty when the data covariance matrix is singular, $|\mathbf{\Sigma}| = 0$, and although the important directional adjoint $Adj(\Sigma)$ exists, the information matrix Σ^{-1} cannot be computed.

2. *Modeling complexity* exists when $1 \leq q < n$, and the integer q = unknown and has to be identified from the (inexact) data. The modeling complexity for a given n number of variables is determined by q and can be measured by the combinatorial expression:

$$\begin{aligned} C(n,q) &= \binom{n}{q} \\ &= \frac{n!}{q!(n-q)!} \end{aligned} \tag{4}$$

The modeling complexity increases rapidly, when n, the number of observed variables in the model, increases. Notice that the modeling complexity is unknown but bounded and computable, for the unknown, but bounded integer q, since $1 \leq q < n$. For example, for $C(n,q) = C(5,2) = 10$, since one can choose in ten different ways a $2-$equation model from the five rows in the information matrix. But already $C = (7,3) = 35$.

Modeling complexity is a symmetric concept: when q is close to 1, or when q is close to n, the model complexity is low. It is highest when $q = n/2$ (for n even), or when $q = \frac{n\pm1}{2}$(for n odd). Intuitively, this can be understood when one realizes that for $q = 1$, the model described a flat plane in the $n-$dimensional data scatter, popularized in the past two hundred years by "single equation regression analysis", while for $q = n - 1$, the model describes a line", popularized in the past hundred years as "factor analysis". Any linear model in between these two extremes describes an much more complex linear system object.

What happens when researchers do not report all possible results of their empirical data investigation? They under-report the number of outcomes. The percentage of analytic incompleteness and the under-reporting of the possible number of projections can be quantified, using the following two measures. First, the information matrix $\mathbf{\Sigma}^{-1}$ is the inverse of the covariance matrix $\mathbf{\Sigma}$ of all n variables in the data set.[1] Each row of the information matrix is an elementary regression or $(n, 1)$ unidirectional LS projection. Since only one of these elementary regressions is reported in each of the following articles, the

$$\text{Percentage of analytic incompleteness} = 100.(n-1)/n\%. \tag{5}$$

Second, the complete number of projections of the invariant number q of possible linear relations among n variables is given by

$$\text{Number of possible LS projections } = \sum_{q=1}^{n-1} C(n, q) \tag{6}$$

The under-reporting is this number minus the one $(n, 1)$ unidirectional projection that is reported in the literature. The following Table 1. provides some examples of these measurements of published scientific incompleteness and bias to demonstrate the seriousness of the problems. These examples are discussed in greater detail in Los (2004).[2]

[PLACE TABLE 1 ABOUT HERE]

3. *Modeling rank* $r = rank(\mathbf{\Sigma}) = rank(\mathbf{H\Phi H'}) = rank(H) = n - q$, with the modeling corank $1 \leq q < n$, and the integer $q = known$. Here $\mathbf{\Phi}$ is a $q \times q$ matrix, $\mathbf{H}$ an $n \times q$ matrix.

We will now illustrate these three rather abstract modeling concepts with three concrete modeling situations in finance. First, by Galton's Error in the conventional computation of the CAPM's

[1] A lagged variable counts as a separate variable.

[2] LS = Least Squares projection (regression);
VAR = Vector Auto-Regression;
ADF = Augmented Dickey-Fuller test

beta of relative risk, used for ranking mutual funds. Second, by the identification of a country model's corank from a series of three financial variables in, for example, a Dividend Discount Model for country risk analysis. Three, by the problem of instability of the portfolio's efficiency frontier, when Markowitz portfolio optimization is combined with exact return and risk attribution.

2 MODEL UNCERTAINTY

The problem of *scientific measurement* is how to identify, or realize, a model, or system, from inexact empirical data. Many disciplines, including finance, use incomplete *unidirectional* projections for the process of system identification, combined with statistical hypothesis testing based on assumed probability. The results of such prejudiced statistical modeling turn out to be biased and unreliable, and they remain disputable. For example, the debate about the two main valuation models in modern asset valuation and portfolio management - the CAPM and APT - has been raging in the financial literature in the past 25 years. The debate heated further up after the severe critique of the CAPM by Fama and French (1992) and no obvious resolution of that debate is in sight.

The crucial question is: why? The answer is: a monumental scientific error made more than hundred years ago by Galton, the inventor of the omnipresent "regression analysis" (Los, 1999, 2001; Kassabov and Los, 2004). What was Galton's Error? Most scientists now acknowledge that it was a serious scientific error of Galton to accept downward biased regression results as conclusive evidence for his asserted hereditary process of "regression towards the mean" of the stature, or height, of the human race. Because, if Galton had correctly interpreted the computational results of what we now call "reverse regressions" (which, surprisingly, Galton did run in both his 1885 and 1886 papers), he could possibly have derived the opposite conclusion: that historically the stature, or height, of the human race becomes more dispersed. From a scientific point of view, the uncertainty of Galton's data should not have allowed him to draw his biased conclusion that the stature of men is diminishing over time, *since his data evidence was too uncertain to be factually* *conclusive*. The variation in Galton's data was 77.8% unsystematic, or uncertain, and only 22.2% systematic, as can be checked using Galton's own published results.

This seemingly innocent practice of benchmarking (and of the related style investment) is not without serious consequences. Among financial regulators there is still an alarming, but misdirected, regulatory interest in a single risk measure to classify mutual funds. Sharpe's beta has been proposed by many analysts as such a universal systematic risk measure. The following sound a clear warning for the financial services industry, in particular the mutual funds industry and its regulators, to distrust its conventionally computed risk measure - Sharpe's beta - and to not base global capital pricing on this prejudiced and biased measure.

2.1 2D Complete LS Projections

It is crucial for the understanding of the new methodology to note that two variables imply two orthogonal LS projections, or in general, that n variables imply n orthogonal LS projections. Let's focus first on the bivariate case and compute symbolically its *two* extreme noise and signal covariance matrices, assuming first no noise in variable 1, $\widetilde{\sigma}_{11} = 0$, followed by no noise in variable 2, $\widetilde{\sigma}_{22} = 0$. The first orthogonal projection gives $\mathbf{A}_1 = \begin{bmatrix} 1 & -\frac{\sigma_{11}}{\sigma_{12}} \end{bmatrix}$, the second $\mathbf{A}_2 = \begin{bmatrix} 1 & -\frac{\sigma_{12}}{\sigma_{22}} \end{bmatrix}$. Using the *CLS Theorem* to compute the two corresponding extreme LS noise matrices $\widetilde{\mathbf{\Sigma}}_1^{LS}$ and $\widetilde{\mathbf{\Sigma}}_2^{LS}$, we can now find that the LS noise resulting from the corresponding projections is

$$\widetilde{\sigma}_{11} = \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}, \text{ when } \widetilde{\sigma}_{22} = 0 \text{ (= the conventional case)} \tag{7}$$

$$\widetilde{\sigma}_{22} = \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}, \text{ when } \widetilde{\sigma}_{11} = 0 \text{ (= the "reverse" case)} \tag{8}$$

This implies that the *percentage* of epistemic uncertainty of the data is independent of the projection direction, since

$$\begin{aligned}\frac{\widetilde{\sigma}_{11}}{\sigma_{11}} &= \frac{\widetilde{\sigma}_{22}}{\sigma_{22}} \\ &= 1 - \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}} \\ &= 1 - \frac{\beta_2}{\beta_1} \\ &= 1 - \rho_{12}^2 \end{aligned} \tag{9}$$

Thus the complete noise covariance matrix for the first orthogonal projection looks like:

$$\widetilde{\mathbf{\Sigma}} = \begin{bmatrix} \widetilde{\sigma}_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

where the model uncertainty variance is assumed to reside in the first variable, since the noise variance of the variable on which is projected is assumed to be $\widetilde{\sigma}_{22} = 0$ and the model coefficients are

$$\mathbf{A}_2 = \begin{bmatrix} 1 & -\beta_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\sigma_{12}}{\sigma_{22}} \end{bmatrix} \tag{11}$$

Or, in more familiar notation,

$$\begin{aligned}\mathbf{A}_2\mathbf{x}' &= \widehat{x}_1 - \widehat{x}_2\beta_2 = 0, \text{ so that} \\ \widehat{x}_1 &= \widehat{x}_2\beta_2 \end{aligned} \tag{12}$$

with the projection coefficient $\beta_2 = -\frac{\sigma_{12}}{\sigma_{22}}$. This lower, "*vertical* " projection is the only one presented in the financial-economic literature for bivariate data sets. But, of course, to be complete, we have, similarly, for the upper or "*horizontal* " projection (similarly normalized on $\mathbf{x}_1$), which is classically known as the "*inverse* " regression, the complete noise covariance matrix:

$$\widetilde{\mathbf{\Sigma}} = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\sigma}_{22} \end{bmatrix} \tag{13}$$

where now all model uncertainty variance is assumed to reside in the second variable, since the noise variance of the variable on which we project is assumed to be $\widetilde{\sigma}_{11} = 0$ and

$$\mathbf{A}_1 = \left[\begin{array}{cc} 1 & -\beta_1 \end{array} \right] = \left[\begin{array}{cc} 1 & -\frac{\sigma_{11}}{\sigma_{12}} \end{array} \right] \tag{14}$$

Or, in more familiar notation,

$$\begin{aligned} \mathbf{A}_1 \widehat{\mathbf{x}}' &= \widehat{x}_1 - \widehat{x}_2 \beta_1 = 0, \text{ so that} \\ \widehat{x}_1 &= \widehat{x}_2 \beta_1 \end{aligned} \tag{15}$$

with the projection coefficient $\beta_1 = -\frac{\sigma_{11}}{\sigma_{12}}$.

Having provided all the ingredients for linear identification from empirical data, the geometric uncertainty relationship for bivariate systems can now be discussed.

2.2 2D Information Contour Ellipses

Affine transformations help to visualize information in data scatter, in particular the affine invariant norm, which produces the information ellipse, or density contour, in 2D data scatter. The (2×2) data covariance matrix $\mathbf{\Sigma}$ of two empirically observed financial-economic variables are as follows:

$$\begin{aligned} \mathbf{\Sigma} &= \left[\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{array} \right] \\ &= \left[\begin{array}{cc} 0.4286 & 0.1184 \\ 0.1184 & 0.0861 \end{array} \right] \end{aligned} \tag{16}$$

with determinant $|\Sigma| = 2.2884 \times 10^{-2}$. Two particular concentric information ellipses, based on the information matrix Σ^{-1} are the following, scaled by two arbitrary constants $c = 60$ and 150 for visualization in Fig. 1:

$$\begin{aligned}
c &= \widehat{\mathbf{x}}\boldsymbol{\Sigma}^{-1}\widehat{\mathbf{x}} \\
&= \begin{bmatrix} \widehat{x}_1 & \widehat{x}_2 \end{bmatrix} \begin{bmatrix} \frac{\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} & -\frac{\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \\ -\frac{\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} & \frac{\sigma_{11}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} \\
&= 3.7625\widehat{x}_1^2 - 10.3478\widehat{x}_1\widehat{x}_2 + 18.7292\widehat{x}_2^2 = 60 \text{ and } 150 \qquad (17)
\end{aligned}$$

These two equations represent the two concentric ellipses, or density contours of the original 2D data scatter, as plotted in Fig. 1.

[PLACE FIG. 1 ABOUT HERE]

The two variable, single equation, $(n,q) = (2,1)$, orthogonal Least Squares projections, with coefficients from the rows of the information matrix, can be found by setting the first derivatives of the information ellipse equal to zero, as follows:

$$\begin{aligned}
\frac{1}{2}\frac{\partial c}{\partial \widehat{\mathbf{x}}} &= \frac{1}{2}\frac{\partial \widehat{\mathbf{x}}\boldsymbol{\Sigma}^{-1}\widehat{\mathbf{x}}}{\partial \widehat{\mathbf{x}}} \\
&= \boldsymbol{\Sigma}^{-1}\widehat{\mathbf{x}} \\
&= \begin{bmatrix} 3.7625 & -5.1739 \\ -5.1739 & 18.7292 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} 3.7625\widehat{x}_1 - 5.1739\widehat{x}_2 \\ -5.1739\widehat{x}_1 + 18.7292\widehat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (18)
\end{aligned}$$

Fig. 1 shows that these two LS projections form a convex cone of two lines in the 2D data scatter space. Each line is determined by the unique orthogonal projection on one of the two data axes. In Fig. 1 these points can be found where the ellipse contours are vertical and horizontal, *i.e.*, where they are parallel to the data axes.

These orthogonal LS projection lines form a convex cone "around" the principal axis of the information ellipse. This becomes obvious when the information ellipse and the two orthogonal LS projection lines (bold) are visualized together in the same data frame of reference, as in Fig. 1, where also the principal and minor axes (thin lines) of the ellipse are drawn in. The principal axis (red) lies in the length of the ellipse, while the minor axis (blue) is orthogonal to the principal axis. The directions of these principal and minor axes are found from the eigenvectors of the information matrix $\mathbf{\Sigma}^{-1}$. The lengths of these axes can be found from the corresponding eigenvalues. One of the orthogonal projection lines - of the "regression" of nominal GDP on the stock index - happens to lie very close to the principal axis and, therefore, appears to be statistically "most acceptable," although that is a prejudice from the perspective of the data ellipse. In fact, the other orthogonal projection line - of the "reverse regression" of the stock index on nominal GDP - does not lie close to the principal axis and would then be labelled "least acceptable." But such a situation isn't determinable *a priori* from the data. Therefore, such *a priori* distinction between "regressand" and "regressor" data variables is scientifically unjustified. It is necessary to take account of the *complete* set of data, *i.e.*, the whole information matrix $\mathbf{\Sigma}^{-1}$. The angle between the two orthogonal projection lines is an indication of the relative uncertainty, measured by the noise/data ratio $\frac{N}{D} = 1 - \rho_{12}^2$, where $\rho_{12}^2 = \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}$ is the coefficient of bivariate determination.

Simple trigonometry shows that for bivariate systems the degree of identification or model determination is given by the conventional coefficient of determination.

$$\rho_{12}^2 = \frac{\beta_2}{\beta_1} = \frac{tan(\theta_1)}{tan(\theta_1 + \theta_2)} \tag{19}$$

where the angles $\theta_1 = atan(\beta_2)$ and $\theta_3 = atan(1/\beta_1)$, with $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2}$ radians.

The angle θ_2 between the cone formed by the two systematic slope lines of the elementary LS measures the finite modeling uncertainty. The true systematic slope coefficient β lies in between these two extreme slopes and is uncertain, *i.e.*, inexact, although it is uniquely computed by a particular LS projection. In principle, there may exist an infinite number of LS projections between the two extreme elementary LS projections. Each of these projections must be a linear combination of these two extremes. The closer the slopes of the two extreme elementary projections are together, the more certain we can be of the model coefficient β.

2.3 2D Modeling Uncertainty Measured

Using the preceding uncertainty relationship and the noise/signal ratio, there are now, at least, five *equivalent* ways of presenting bivariate modeling certainty and uncertainty:

(1) Bivariate modeling certainty

(i) $|\mathbf{\Sigma}| = 0$, the data covariance matrix is singular, *i.e.*, its determinant equals zero;

(ii) $\rho_{12}^2 = 1$, the coefficient of bivariate determination shows complete explanation or exact determination;

(iii) $\beta_2 = \beta = \beta_1$, the upper and lower projection slopes do coincide;

(iv) $\theta_2 = 0$, there exists no uncertainty gap in between the orthogonal frames of data reference;

(v) $\frac{N}{S} = 0$, the noise/signal ratio is zero, since the exact data consist only of the signal.

(2) Bivariate modeling uncertainty

(i) $|\mathbf{\Sigma}| > 0$, the data covariance matrix is positive definite, *i.e.*, its determinant is positive;

(ii) $0 < \rho_{12}^2 < 1$, the coefficient of bivariate determination shows less than complete explanation or inexact determination;

(iii) $\beta_2 < \beta < \beta_1$, the upper and lower projection slopes do not coincide;

(iv) $0 < \theta_2 < \frac{\pi}{2}$, there exists an uncertainty gap in between the orthogonal frames of data reference;

(v) $\frac{N}{S} > 0$, the noise/signal ratio is positive, since the inexact data contain some noise, together with the signal.

2.4 Empirical Example: Beta-Based Mutual Funds Selection

Despite the early recognition of Galton's error, the statistical literature, including the financial literature, still reports exclusively the lower projection slope β_2 and the bivariate coefficient of determination ρ_{12}^2, but not the upper projection slope β_1. Also, it doesn't report the noise/signal ratios, *i.e.*, ratio of the unsystematic risk to the systematic risk. In other words, it reports only the downward biased computational result of β, often, but not always, together with an indication of the model uncertainty ρ_{12}^2, but it does not provide the complete picture. This deficiency is even more pronounced for the cases with more than two variables, where it is never reported how the system invariant q is determined, otherwise than from "theory." In almost all cases, it is (incorrectly) assumed that $q = 1$, the model consists of a single linear equation, *i.e.*, a "plane" in the, nonconforming mdmv scatter data space.

Current financial industry presentation standards recommend to select mutual funds by their funds by their risk/return profile. The systematic risk is measured by the relative rate of return volatility, *i.e.*, as covariance risk measured relative to that of a benchmark market index. The return is measured by some average return over a appropriate time period. The relative risk measure is called Sharpe's (1963) "beta". When a fund's beta, β, is below unity, the fund is categorized as "defensive," because the volatility of its investment returns is lower than that of the market as a whole. With a β greater than unity, a fund is categorized as "aggressive." Finally, with a β equal to unity, the fund is classified as a neutral, or a market-index-like, fund, because it behaves similarly to the selected market index.

Sharpe's beta is computed and presented by the financial industry as the lower projection β_2, as recommended, for example, by *The AIMR Performance Presentation Standards* of 1993 and 1996 (AIMR, *1993*, pp. 34 - 35, and AIMR, 1996, pp. 92 - 95) which are adopted as part of the AIMR's Standard of Professional Conduct V.B concerning Performance Presentation. The deficient, but official recommendation concerning the computation and presentation of the beta is now promoted to become a global standard [3] . These simple computations lead to a severe under-representation of the empirically observed systematic risks of the selected funds by the financial industry. Therefore, the question can be rased if the recommendations by the AIMR are consistent with its own Standard of Professional Conduct IV, the Relationships with and Responsibilities to Clients and Prospects, in particular Standard IV.A.2 concerning Research Reports and Standard IV.A.3 concerning Independence and Objectivity.

This under-representation of systematic investment risk can be demonstrated by looking at how many mutual funds are ranked aggressive, defensive, or equivalent to the market index by Sharpe's beta and how many are truly aggressive defensive or neutral, when taking account of all the modeling uncertainty implied by the data. For the data we use the computed betas and coefficients of determination in Morningstar's convenient (Windows based) *Principia for Mutual Funds* of July 1995, as released on computer diskettes to the public on December 31, 1995.[4]

[PLACE TABLE 2 ABOUT HERE]

First, we notice in Table 2. that only $3,227$ out of a total universe of $7,051$ funds have measurable risk, as indicated by a computed coefficient of determination larger than zero, or 45.8% of the total universe. The other funds are younger than 3 years and don't have a 3-year record to base such computations on. However for 12 of these $3,227$ funds the lower beta β_2 equals

[3] The original AIMR Performance Presentation Standards (AIMR, 1993), which took effect on January 1, 1993, were amended and restated on September 13, 1996 to include some international concerns (AIMR, 1996a). This restatement did not amend the incomplete computation of Sharpe's beta. The AIMR Performance Presentation Standards form part of the AIMR's Code of Ethics and Standards of Professional Conduct (AIMRb). They're also part of the newly proposed Global Investment Performance Standards (AIMR, 1998). Furthermore, the rating agencies compute and publish (lower bound) betas for stocks, e.g. in *Standard & Poor's Stock Reports* (Standard & Poor's Corp., New York) and *Value Line Investment Survey* (Value Line Publishing, Inc., New York). But their reports do not include the bivariate coefficient of determination, without which the epistemic uncertainty, and thus the upper bound betas, can't be ascertained. Thus their reports do not accurately present the measured data uncertainty. In 2003 the Association for Investment Management and Research has changed its name to The CFA Institute.

[4] These data diskettes are available, at cost, from Morningstar, Inc., 225 West Wacker Drive, Chicago, Illinois 60606, and are updated quarterly. Morningstar is a respected mutual funds monitor with an excellent reputation that computes the betas and corresponding coefficients of determination of the mutual funds strictly according to the accepted industry standards. According to Morningstar's OnFloppy User's Guide (p.22): "Morningstar bases alpha, beta, and R^2 on a least squares regression of the fund's excess return over T-bills compared with the excess returns of the fund's benchmark index. These calculations are computed for the trailing 36-month period."

zero in the two published digits beyond the decimal point. Thus only $3,215$ funds have measured systematic market risk as defined by the CAPM, or 45.6% of the total universe.

If we accept Sharpe's criterion for selecting funds by their relative volatility or systematic market risk characteristic, then the number of defensive funds selected by correctly implementing Sharpe's beta is 25.6% of the $2,047$ claimed to be defensive by the current industry standards. In addition, the number of actual market index funds is only 26.9% of the 67 funds claimed to be market index funds in this representative data universe. Finally, of the $3,215$ funds for which the appropriate data were available 954, or 45%, could not be categorized as defensive, aggressive or market index, in spite of the claims of the financial industry.

To gain an impression of some of the investment magnitudes involved, look at the following figures. The mutual fund industry in the United States grew from US95 billion in assets in 1979, to nearly US2 trillion by the end of 1994, an increase of over 20 times. Even after taking account of consumer price inflation and the resulting loss of purchasing power in the U.S. of more than 90% over the same period, that is still a very sizeable increase in real assets of eleven times in fourteen years.

Most of this increase has actually occurred in the last three years. American investors poured a net US377 billion into equity mutual funds alone in $1993-95$. Since the end of 1994 until the middle of 1996, the Dow Jones industrial average climbed by nearly 50% and the broader $S\&P500$ index by 46%, increasing America's financial wealth by US2.4 trillion, more than the entire annual output of Germany.[5]

Compare now these market sizes with the magnitudes of the universes we analyzed. By September 1993 there existed $4,347$ open-ended mutual funds. The following year Morningstar monitored about 79% of them. Its *Mutual Funds OnFloppy* universe contained $3,434$ funds with an average median market capitalization of US0.5 billion in net assets by the end 1994. Its updated successor universe, Morningstar's *Principia for Mutual Funds,* used in our analysis, contained already

[5] *The Economist*, July 6, 1996, pp. 18 and 21.

more than double this number at the end of 1995: $7,051$ funds. Because of the fast growth in the number of new funds, there were now many more smaller funds include, since the average median market capitalization of this universe is US\$264.9 million in net assets. But the more restricted universe of $3,215$ funds, on which the conclusions of Table 2. are based, has an comparable average median capitalization of US\$514.6 million in net assets, while the universes of 450 funds and of 81 funds have average median market capitalizations of US\$510.5 million, respectively US\$510.4 million in net assets.

Since this increasingly massive process of mutual fund selection and pricing is biased by the under-representation of market risk, very serious misallocation between the investment alternatives could result, based on their currently presented biased relative risk and return profiles. Also, since a substantial amount of this investment may be hot, these market allocations are not likely to be patient or secure. Indeed, *The Economist* refers to the argument "that many mutual-fund investors do not understand what they are doing; and that, when they realize what the risks are, they will flee."[6] There is no reason for panic, however, according to the same article, because of the apparent maturity of the modern investors. The younger investors "not only say they accept the risk involved - in a recent survey by American financial regulators, 94% of investors said they knew they could lose money in share dealings as well as gain it - they also seem, in practice, to respond calmly when prices fall."

The biased published betas do not only raise macro concerns relevant for national policy makers or global asset allocators, but also micro concerns relevant for individual portfolio managers. This paper suggests that there is more uncertainty about the systematic risk than current portfolio managers, regulators and the educators of financial analysts recognize. A scientific debate on the issue of the adequacy of a single risk measure for mutual funds, like the beta, is therefore timely. The Securities and Exchange Commission (SEC), in reaction to recent sharp price drops for several supposedly low-risk mutual funds, has asked fund managers to look more carefully at

[6] *The Economist*, July 6, 1996, p. 18.

their risk management controls that track derivative positions.[7] The SEC is trying to condense the myriad risks of mutual funds into a single measure that would convey these risks to investors.[8] In 1995 the SEC floated a Concept Release (= White Paper) on the issue, requesting comments on or before July 7, 1995. The comments in this paper should forewarn the SEC that its quest for a single measure for multi - faceted investment risk is likely to be just as quixotic and fruitless as the quest for a single IQ measure, when the fundamental principles of science are ignored.

A complete representation of the empirical systematic uncertainty and risk is required [9] . Thus for the bivariate CAPM two measures must be published: the correlation coefficient ρ_{12} (or, equivalently in the bivariate case, the coefficient of determination ρ_{12}^2) together with the beta β_2, since all other bivariate measurements can be derived from these two. Next, one must educate the investors about the uncertainty range for β, about $\beta_2 \leqslant \beta \leqslant \beta_1 = \frac{\beta_2}{\rho_{12}^2}$. It was because of the recommendable practice of Morningstar to publish both β_2 and ρ_{12}^2 that we were able to properly categorize the mutual funds, while still using the accepted industry standard of CAPM style taxonomy.

3 MODEL COMPLEXITY

The international fund services of CITCO advertised in 1997 that "Complexity is a multi-headed monster that can wreak havoc on investors, companies and institutions, who have assets to protect, preserve and enhance. " In the context of financial and economic modeling, the concept of (linear) system complexity is usually dealt with by presuming to know *a priori* the model's corank q, *i.e.*, the model rank $r = rank(\widehat{\Sigma})$, as the Cowles Commission/Foundation erroneously prescribed in

[7] Hansell, S., "S.E.C. Asked to Study Derivatives in Funds," *The New York Times*, June 16, 1994, D8.

[8] Hansell, S., "U.S. Seeking Mutual Fund Gauge: Wants a simple system to inform investors," *The New York Times*, June 20, 1994, D1 - D2.

[9] Of course, an investor can reduce the risks of his portfolio further by appropriate diversification, as Markowitz demonstrated in 1952. The current paper only adds that, while Sharpe's erroneous beta compares with Galton's error of regressing towards the mean, the current practice of factor, or principal components analysis of investment portfolios, based on APT compares more closely to the erroneous practice of IQ testing. The current paper follows deliberately Sharpe's 1963/64 CAPM approach to mutual fund selection, since that is still the most widely accepted and recommended standard in the financial industry.

the early 1950s. This presumed model rank is then superimposed on the data when we "estimate "the reduced form simultaneous equation model. " However, this model rank can be identified and all corresponding (Grassmanian) coefficients of the model can be computed, by Complete LS Projection, from only the data. Complete LS Projection means here: LS projection on all data series, *i.e.*, in all basic directions. This will now be illustrated by the identification of the model rank of a simple financial system in Taiwan.

Thus the main question for global investors is: what is the systematic relationship between these countries' stock market valuations, their nominal GDPs and their (short term) interest rates?

In this paper, we use a new system identification or "superfiltering" methodology to answer this question.[10] This superfilter methodology extracts the financial economic system structures from the empirical observations without undue theoretical presumptions. The country's major stock market index is used to track the valuation of its stock market. Its nominal GDP and short term interest rate are the two other macroeconomic variables. The data used for this analysis range from 1986 first quarter to 1995 third quarter.

For the analysis we combine some elements of the conventional Dividend Discount Model (DDM) and of modern Asset Pricing Theory (APT), but, taking account of some earlier critical discussions in, we allow for greater parameter freedom and we discuss the advantages and disadvantages of principal component versus projection theory for multivariate analysis.

First, inspired by multivariate Arbitrage Pricing Theory (APT) and the classical economists' Cobb-Douglas (C-D) Production Function Theory, we postulate the following single equation ($q = 1$) system structure for the "modified" DDM

$$\ln S_t = a.\ln NGDP_{t+1} + b.\ln IR_t + \ln D \tag{20}$$

Effectively, we introduce two elasticities, whose (uncertain) values are to be determined from the

[10] The designation "superfilter" for our new methodology was first used by Professor Rouchaleau of the École des Mines, Paris, at an advanced econometrics seminar at I.N.R.E.A. in Sophia Antipolis in Southern France in the Spring of 1996, according to a personal e-mail message of March 26, 1996 from Professor Kalman of the ETH, Zürich. For the theoretical foundation of the superfilter, see [?], [?],[?], [?] and [?].

empirical data:

> a = nominal GDP ("expected income") elasticity, which finance theory expects to be positive; and b = interest rate elasticity, which finance theory expects to be negative.

Theoretically, the added $\ln D$ term represents the deterministic value innovations in the stock market introduced by technological advances. By exponentiation the modified DDM model transforms to the flexible nonlinear structure

$$S_t = D.(NGDP_{t+1})^a.(IR_t)^b \tag{21}$$

Notice that this structure encompasses the original DDM structure, when the theoretical parameters $a = 1$, $b = -1$ and $D = 1$.

Second, we introduce further structure flexibility by allowing for two independent linear equations ($q = 2$), by postulating the following system:

$$\ln S_t = c. \ln NGDP_{t+1} + \ln F \tag{22}$$

$$\ln S_t = d. \ln IR_t + \ln G \tag{23}$$

where c and d are the new expected income and interest rate elasticities respectively. This two equation system encompasses the single equation system ($q = 1$) by linear combination, *i.e.*, by taking a weighted average of the two independent equations.

While we can create a unique single equation system from this two-equation system, the reverse does *not* hold true. Thus single ($q = 1$) and two-equation ($q = 2$) systems are structurally *not* equivalent. In terms of principal component analysis, a two-equation system ($q = 2$) behaves like a one factor ($r = 1$) system, since all variables move simultaneously as a bundle in the same (or opposite) direction. In contrast, in a true single equation system ($q = 1$) there are two factors ($r = 2$), since two variables move independently from each other.

Note that $r+q = n$ where n is the number of *variables*, q is the number of *independent equations* and r is the number of *factors*. In our analysis, $n = 3$ for each of the six Asian countries. Since the $q = 2$ system encompasses the $q = 1$ system, the $q = 2$ system forms the basis for our system identification procedure to determine which of the two system structures provides the best explanation of the observed noise-contaminated empirical data covariances.

By exponentiation, this C-D model transforms to the system of two simultaneous *nonlinear* equations:

$$S_t = F.(NGDP_{t+1})^c \tag{24}$$

$$S_t = G.(IR_t)^d \tag{25}$$

The covariance analysis to determine the system structure (is $q = 1$ or 2 ?) and the finite parameter ranges for the elasticity parameters, $c_* \leq c \leq c^*$ and $d_* \leq d \leq d^*$, is executed on laterally shifted frames of data reference, as follows:

$$x_{1t} = \ln S_t - \overline{\ln S_t} \tag{26}$$

$$x_{2t} = \ln NGDP_{t+1} - \overline{\ln NGDP_{t+1}} \tag{27}$$

$$x_{3t} = \ln IR_t - \ln IR_t \tag{28}$$

The averages, indicated by bars over the variables, are taken over all $T = 39$ observations ($T = 38$ for Indonesia). After the covariance analysis the (projected and non-projected) deviations are transformed into the original variables by adding back these averages. For example, $\ln S_t = x_{1t} + \overline{\ln S_t}$. By taking averages the "residual technology" terms, $\ln F$ and $\ln G$ are of no substantial importance for the system identification. Once the parameter ranges for the income elasticity c and the interest elasticity d are computed, the parameter ranges for the $\ln F$ and $\ln G$ terms (respectively for F and G) can immediately be determined.

3.1 3D Complete LS Projections

We can combine this bivariate projection information in the form of 3D Complete Least Squares projections for both $q = 2$ and $q = 1$. The $q = 2$ CLS plots encompass the $q = 1$ plots. The

$q = 2$ plots are *rays*, representing the projected $q = 2$ systems, while the $q = 1$ systems are *planes*, representing the projected $q = 1$ systems.

It can be shown that the three $(n, q) = (3, 2)$ CLS systematic projector matrices

$$\widehat{\mathbf{P}}_i = \widehat{\mathbf{\Sigma}}^{CLS}\mathbf{\Sigma}^{-1} \text{ with } i = 1, 2, 3 \tag{29}$$

are configured as follows.[11]

For the $q = 2$ projections on variable x_1 we have the systematic signal projector

$$\widehat{\mathbf{P}}_1^{CLS} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\sigma_{12}}{\sigma_{11}} & 0 & 0 \\ \frac{\sigma_{13}}{\sigma_{11}} & 0 & 0 \end{bmatrix} \tag{30}$$

This signal projector, $\widehat{\mathbf{P}}_i^{CLS}$ combines the bivariate covariances of the data covariance matrix Σ. When $\widehat{\mathbf{P}}_1$ is post-multiplied by the $n \times T = 3 \times 39$ data matrix $\mathbf{x}' = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, so that the 3×39 systematic data matrix is $\widehat{\mathbf{x}}'_1 = \widehat{\mathbf{P}}_1^{CLS}\mathbf{x}'$, the first data series in $\widehat{\mathbf{x}} = (\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \widehat{\mathbf{x}}_3)$, remains non-projected, $\widehat{\mathbf{x}}_1 = \mathbf{x}_1$, since it is the series on which we project, *i.e.*, the one which is assumed to have no noise, $\widetilde{\sigma}_{11} = 0$. The other two series, $\widehat{\mathbf{x}}_2$ and $\widehat{\mathbf{x}}_3$ result from the simple bivariate orthogonal projections on $\mathbf{x}_1$. Thus we have the first CLS_1 projected system

$$\begin{aligned} \widehat{\mathbf{x}}_1 &= \mathbf{x}_1 \\ \widehat{\mathbf{x}}_2 &= \tfrac{\sigma_{12}}{\sigma_{11}}\widehat{\mathbf{x}}_1 \\ \widehat{\mathbf{x}}_3 &= \tfrac{\sigma_{13}}{\sigma_{11}}\widehat{\mathbf{x}}_1 \end{aligned} \quad \text{which is equivalent to} \quad \begin{aligned} \widehat{\mathbf{x}}_1 &= \mathbf{x}_1 \\ \widehat{\mathbf{x}}_1 &= \tfrac{\sigma_{11}}{\sigma_{12}}\widehat{\mathbf{x}}_2 = c.\widehat{\mathbf{x}}_2 \\ \widehat{\mathbf{x}}_1 &= \tfrac{\sigma_{11}}{\sigma_{13}}\widehat{\mathbf{x}}_3 = d.\widehat{\mathbf{x}}_3 \end{aligned} \tag{31}$$

Notice that this projection system consists of two planes which cut each other and form a ray through the origin of the data frame of reference. Each ray provides a bit of information about the true underlying system, but only when viewed together with the rays from the other two orthogonal projections. In fact, an infinite number of projections - a *complete projection cone* - can be computed from linear combinations from these three extreme orthogonal CLS projections.

[11] Hats $\widehat{..}$ denote systematic signals and waves $\widetilde{..}$ denote unsystematic noise. For simple algebraic derivations of the $(3, 2)$ and $(3, 1)$ systematic projector matrices $\widehat{\mathbf{P}}_i$, see Appendices II and IV.

Similarly, the other two extreme $(n, q) = (3, 2)$ systematic projectors for the CLS_2 and CLS_3 projections are given by

$$\widehat{\mathbf{P}}_2^{CLS} = \begin{bmatrix} 0 & \frac{\sigma_{12}}{\sigma_{22}} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\sigma_{23}}{\sigma_{22}} & 0 \end{bmatrix} \tag{32}$$

and

$$\widehat{\mathbf{P}}_3^{CLS} = \begin{bmatrix} 0 & 0 & \frac{\sigma_{13}}{\sigma_{33}} \\ 0 & 1 & \frac{\sigma_{23}}{\sigma_{33}} \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

3.2 3D Information Contour Ellipsoid

Affine transformations help to visualize information in 3D data scatter, in particular the affine invariant norm, which produces the information ellipsoid, or 3D density contour map, in 3D data scatter. One particular information ellipsoid based on the (3×3) covariance matrix $\mathbf{\Sigma}$ of three empirically observed financial economic variables for Taiwan - the natural logarithms of its stock market index, its nominal GDP (one-quarter-ahead) and its bank lending rate for the period 1986Q1 - 1995Q3 - is the following, scaled by the (arbitrary) constant $c = 100$:

$$\begin{aligned} c &= \widehat{\mathbf{x}}\mathbf{\Sigma}^{-1}\widehat{\mathbf{x}} \\ &= \begin{bmatrix} \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3 \end{bmatrix} \begin{bmatrix} 4.7529 & -4.3230 & -3.4514 \\ -4.3230 & 19.4717 & -2.9773 \\ -3.4514 & -2.9773 & 12.0409 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix} \\ &= 4.7529\widehat{x}_1^2 - 8.6460\widehat{x}_1\widehat{x}_2 - 6.9028\widehat{x}_1\widehat{x}_3 + 19.4717\widehat{x}_2^2 - 5.9546\widehat{x}_2\widehat{x}_3 + 12.0409\widehat{x}_3^2 = 100 \end{aligned} \tag{34}$$

This quadratic equation represents an ellipsoid - a "cigar" - in the 3D data variable space, as can be seen in Fig. 2. The volume of this information ellipsoid is determined by the arbitrary constant c. Otherwise stated, this information ellipsoid, or 3D data scatter density map, represents the data scatter density of 3D contour level c. Since c is arbitrary, other 3D contour levels, *i.e.*, concentric ellipsoids of different volumes, could be plotted.

[PLACE FIG. 2 ABOUT HERE]

The three 3-variable, single equation, $(n, q) = (3, 1)$, orthogonal Least Squares projections, with coefficients from the rows of the information matrix $\mathbf{\Sigma}^{-1}$, can be found by setting the first derivatives of the information ellipsoid equal to zero, as follows:

$$\begin{aligned}
\frac{1}{2}\frac{\partial c}{\partial \widehat{\mathbf{x}}} &= \frac{1}{2}\frac{\partial \widehat{\mathbf{x}}\mathbf{\Sigma}^{-1}\widehat{\mathbf{x}}}{\partial \widehat{\mathbf{x}}} \\
&= \mathbf{\Sigma}^{-1}\widehat{\mathbf{x}} \\
&= \begin{bmatrix} 4.7529\widehat{x}_1 - 4.3230\widehat{x}_2 - 3.4514\widehat{x}_3 \\ -4.3230\widehat{x}_1 + 19.4717\widehat{x}_2 - 2.9773\widehat{x}_3 \\ -3.4514\widehat{x}_1 - 2.9773\widehat{x}_2 + 12.0409\widehat{x}_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{35}$$

Fig. 2 shows that these three LS projections form a convex cone of three planes in the 3D data scatter space. Each plane is determined by the orthogonal projection on two of the three data axes.

These orthogonal LS projection planes form a convex cone "around" the principal axis of the information ellipsoid. The principal axis lies in the length of the ellipsoid, through its center. The directions of the principal and minor axes of the ellipsoid can be found from the eigenvectors of the information matrix $\mathbf{\Sigma}^{-1}$. The length of these axes can be found from the absolute value of the corresponding eigenvalues

In this case the orthogonal projection planes happen to almost "rotate" around the principal axis, providing strong evidence that a proper linear model for the data scatter would have a $(n, q) = (3, 2)$, instead of an $(n,) = (3, 1)$ configuration. Each projection plane represents a single equation, $(n, q) = (3, 1)$, projection of one variable, the "regressand," on the other two. Fig. 2 clearly demonstrates that an *a priori* distinction between "regressand" and "regressor" data variables is scientifically unjustified, since each plane lies in a different direction, representing only part of the data.. It is necessary to take account of the *complete* set of data, *i.e.*, the whole information matrix Σ^{-1}.

The volume of the cone formed by the three orthogonal projection planes is an indication of the relative uncertainty, measured by the data ratio

$$\frac{N}{D} = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23} \tag{36}$$

where ρ_{ij} is the bivariate correlation coefficient between variables i and j.

3.3 3D Modeling Uncertainty Measured

Analogously to the bivariate case, we have the following equivalent ways of presenting modeling certainty and uncertainty for the trivariate case:

(1) Trivariate modeling uncertainty

(i) $|\mathbf{\Sigma}| > 0$, the data covariance matrix is positive definite, *i.e.* its determinant is positive;

(ii) $0 < \rho_{ij}^2 < 1$, for some $i, j = 1, 2, 3, i \neq j$, *i.e.*, some coefficient of bivariate determination shows less than complete explanation or inexactness;

(iii) $\widehat{\mathbf{P}}_i^{CLS} \neq \widehat{\mathbf{P}}_j^{CLS}$(after normalization), for all $i, j = 1, 2, 3, i \neq j$, the CLS projectors don't coincide;

(iv) $0 < \left|\widehat{\mathbf{P}}_1^{CLS} + \widehat{\mathbf{P}}_2^{CLS} + \widehat{\mathbf{P}}_3^{CLS}\right| < 1$, there exists an uncertainty gap within the orthant frame of data reference;

(v) 3D $N/S > 0$, the noise/signal ratio is positive, since the inexact data contain some noise together with the signal.

(2A) Trivariate modeling certainty for q=2

(i) $|\mathbf{\Sigma}| = 0$, the data covariance matrix is singular, *i.e.*, its determinant equals zero;

(ii) $\rho_{ij}^2 = 1$, for *all* $i, j = 1, 2, 3, i \neq j$, *i.e.*, *all* coefficients bivariate determination show exactness;

(iii) $\widehat{\mathbf{P}}_i^{CLS} = \widehat{\mathbf{P}}_j^{CLS}$(after normalization), for all $i, j = 1, 2, 3, i \neq j$, *all* CLS projectors coincide;

(iv) $\left|\widehat{\mathbf{P}}_1^{CLS} + \widehat{\mathbf{P}}_2^{CLS} + \widehat{\mathbf{P}}_3^{CLS}\right| = 0$, there exists no uncertainty gap within the orthant frame of data reference;

(v) 3D $N/S = 0$, the noise/signal ratio is zero, since the data contain only the signal.

(2B) Trivariate modeling certainty for q=1

(i) $|\mathbf{\Sigma}| > 0$, the data covariance matrix is positive definite, *i.e.*, its determinant is positive definite;

(ii) $\rho_{ij}^2 = 0$, for some $i, j = 1, 2, 3, i \neq j$, *i.e.*, at least one coefficient of bivariate determination shows no explanation;

(iii) $\widehat{\mathbf{P}}_i^{CLS} \perp \widehat{\mathbf{P}}_j^{CLS}$ and $\widehat{\mathbf{P}}_i^{CLS} \neq \widehat{\mathbf{P}}_k^{CLS} = \widehat{\mathbf{P}}_j^{CLS}$(after normalization), for some $i, j, k = 1, 2, 3, i \neq j, k$, at least one CLS projectors is orthogonal to another while the other two coincide;

(iv) $0 < \left|\widehat{\mathbf{P}}_1^{CLS} + \widehat{\mathbf{P}}_2^{CLS} + \widehat{\mathbf{P}}_3^{CLS}\right| << 1$, there exists an uncertainty gap within the orthant frame of data reference;

(v) 3D $N/S = 0$, the 3D noise/signal ratio is positive, siince the inexact data contain some noise together with the signal.

3.4 Empirical Example: Financial Risk System Identification

In Fig. 3. we have plotted Taiwan's measured data $\mathbf{x}$ in the 3D data space: x_1 is measured on the first horizontal axis, x_2 on the second horizontal axis and x_3 on the vertical axis. From the 3D scatter plot it is difficult to obtain a definite conclusion although some information may be gleaned from the three 2D scatter plots in the side panels. However, when we plot the three $q = 2$

projection systems $\widehat{\mathbf{x}}_i' = \widehat{\mathbf{P}}_i^{CLS}\mathbf{x}'$, $i = 1, 2, 3$, three rays are produced in the center of the 3D scatter plot, as shown in the same frame of reference (but without the data scatter) in Fig. 4.

[PLACE FIG. 3 AND FIG. 4 SIDE BY SIDE ABOUT HERE]

With appropriate visualization software we can rotate these $q = 2$ systems and we observe that the three CLS rays are lying close together in a fairly tight cone. There is a positive systematic relationship between x_1 and x_2 as observed in the bottom (x_1, x_2) grid in Fig. 4, a positive relationship between x_1 and x_3 as observed in the (x_1, x_3) grid, and consequently, there is also a positive relationship between x_2 and x_3 as observed in the (x_2, x_3) grid. Thus we find that Taiwan has a $q = 2$ financial economic system, represented by the two simultaneous equations

$$\begin{aligned} \widehat{x}_1 - c.\widehat{x}_2 = 0 \quad &\text{or, equivalently} \quad \widehat{x}_1 = c.\widehat{x}_2 \\ \widehat{x}_1 - d.\widehat{x}_3 = 0 \quad &\text{or, equivalently} \quad \widehat{x}_1 = d.\widehat{x}_3 \end{aligned} \tag{37}$$

with both $c > 0$ and $d > 0$ The stock market has a positive income elasticity and a *positive* interest rate elasticity, empirically contradicting conventional *economic* theory, which presumes a negative interest rate elasticity, $d < 0$.

The three finite boundaries of these elasticities from the corresponding bivariate elements in the $\widehat{\mathbf{P}}_i^{CLS}$, $i = 1, 2, 3$ systematic projector matrices are given in Table 3.[12]

[PLACE TABLE 3 ABOUT HERE]

Notice, first, that the wide modeling uncertainty ranges reflect the high noise environments, and, second, that theoretically expected unit elasticities are mostly outside these empirical ranges, contradicting conventional *financial* (DDM) theory.

In Taiwan the noise cone spanned by the three projected rays occupies 37% of the data space. This means that 37% of the 3D variation of the Taiwanese data is unsystematic, while the remaining 63% is systematic. Note that in the case of complete certainty, *i.e.*, when all variation is

[12] $(c, d)_i$ is the set of income (c) and interest elasticities (d) from the $q = 2$ CLS projection on variable $\mathbf{x}_i$.

certain, the three projected rays would coincide on one ray and the 3D noise/data ratio would be zero.

Taiwan's 3D noise/data ratio of 37% implies a 3D noise/signal ratio of 58%, *i.e.*, the unsystematic variation in its financial economy is slightly larger than half of its systematic variation. Thus Taiwan forms a fairly coherent economy. There is interaction between both its financial market and its stock market; between its financial market and its expected economy and, consequently, also between its expected economy and its stock market. But this empirically observable interdependence is neither according to conventional economic, nor according to conventional financial theory.

Our identification results for the Grassmanian invariants using various analytic techniques are summarized in Table 4, where we present our conclusions about structure and relative noise levels.

[PLACE TABLE $ ABOUT HERE]

4 MODEL RANK

Often it has been thought that there is an incompatibility between Markowitz 'portfolio optimization and exact and complete cash flow accounting, because of the resulting singularity of the covariance matrix of the portfolio returns. Or, if the matrix was inverted anyway, because of the resulting instability of the resulting information matrix and of the efficiency frontier. Many professional portfolio managers have complaint that they could not use Markowitz optimization, because it would result in too fast and too radical portfolio allocation shifts. The crux of this problem of exact risk attribution is formed by the rank constraints imposed by the portfolio accounting identities. However, these same constraints provide the solution for this instability problem, since they are the exact selectors of the portfolio allocations. This is illustrated by the optimization of a single portfolio of the exactly attributed multi-currency investment strategies of three and later nine countries (Los, 1998, 2001, 2002; Kassabov and Los, 2004).[13]

At time t an investor has three possible investment instruments: (1) investment in an asset in country i, e.g., a stock or a bond, with rate of return $r_i(t)$, (2) a cash swap with rate of return $c_j(t) - c_i(t)$, with $c_j(t)$ the cash rate in country j into which the nominal is swapped, and $c_i(t)$ the cash rate in country i out of which the nominal is swapped and (3) the foreign currency (foreign currency) appreciation rate $\varepsilon_j(t)$ of country j. Thus one particular bilateral investment strategy at time t is represented by the strategic rate of return

$$s_{ij}(t) = r_i(t) + [c_j(t) - c_i(t)] + \varepsilon_j(t) \tag{38}$$

Such a strategy return is equivalent to the sum of a risk premium and a cash return, *i.e.*, the local market i risk premium $[r_i(t) - c_i(t)]$ and the cash return on currency j, $[c_j(t) + \varepsilon_j(t)]$:

$$s_{ij}(t) = [r_i(t) - c_i(t)] + [c_j(t) + \varepsilon_j(t)] \tag{39}$$

This is also equivalent to the sum of a local market i return, the return on a currency forward cross hedge and the foreign currency j appreciation rate, since

$$\begin{aligned} s_{ij}(t) &= r_i(t) + [\{c_1(t) - c_i(t)\} - \{c_1(t) - c_j(t)\}] + \varepsilon_j(t) \\ &= r_i(t) + [f_i(t) - f_j(t)] + \varepsilon_j(t) \\ &= r_i(t) + f_{ij}(t) + \varepsilon_j(t) \end{aligned} \tag{40}$$

The return on a currency forward is $f_i(t) = c_1(t) - c_i(t)$, with $c_1(t)$ the cash return of the base currency. The return on a currency forward cross hedge $f_{ij}(t)$ consists of the difference between the return on the long domestic forward $f_i(t)$ and the return on the short foreign forward $f_j(t)$. An $n \times n$ non-symmetric strategy matrix at time t is a matrix containing all n^2 bilateral investment strategies

$$\mathbf{S}(t) = \{s_{ij}(t); i, j = 1, ..., n\} \tag{41}$$

[13] The Union Bank of Switzerland (UBS) Cash Overaly Unit in London, UK, uses currently this scheme and refers its cash overlay clients to my papers (Los, 1998, 2001, 2002).

Returning to the rates of return, the strategy matrix $S(t)$ at time t can now be generalized, by using matrix algebra, as follows

$$\begin{aligned}\mathbf{S}(t) &= [\mathbf{r}(t) - \mathbf{c}(t)]\boldsymbol{\iota}_n' + \{[\mathbf{c}(t) + \boldsymbol{\varepsilon}(t)]\boldsymbol{\iota}_n'\}' \\ &= [\mathbf{r}(t) - \mathbf{c}(t)]\boldsymbol{\iota}_n' + \boldsymbol{\iota}_n[\mathbf{c}(t) + \boldsymbol{\varepsilon}(t)]',\end{aligned} \tag{42}$$

where $\mathbf{r}(t)$, $\mathbf{c}(t)$ and $\boldsymbol{\varepsilon}(t)$ are $n \times 1$ data vectors of asset rates, cash rates and foreign currency appreciation rates at time t, respectively, and $\boldsymbol{\iota}_n'$ is a $1 \times n$ unit vector, *i.e.*, a vector consisting of n units, $\boldsymbol{\iota}_n' = [1, 1, ..., 1]$.

Next, the 3-dimensional $n \times n \times T$ historical investment strategy *array* $\mathbf{S}$ is the *sequence* of such strategy matrices $\mathbf{S} = \{S(t); t = 1, ..., T\}$. However, covariance risk analysis with three-dimensional arrays is very difficult and it is easier when these arrays are translated into simpler two-dimensional arrays by vectorization. The proper vectorization. of this strategy array is:

$$\begin{aligned}\mathbf{VEC(S)} &= [vec(\mathbf{S}(1)), vec(\mathbf{S}(2)), ..., vec(\mathbf{S}(T))] \\ &= [\boldsymbol{\iota}_n \otimes \mathbf{I}_n][\mathbf{r} - \mathbf{c}] + [\mathbf{I}_n \otimes \boldsymbol{\iota}_n][\mathbf{c} + \boldsymbol{\varepsilon}] \\ &= \mathbf{H}\begin{bmatrix} [\mathbf{r} - \mathbf{c}] \\ [\mathbf{c} + \boldsymbol{\varepsilon}] \end{bmatrix}\end{aligned} \tag{43}$$

where $\mathbf{r}$ is the $n \times T$ matrix of T observations on the rates of return of n country assets, $\mathbf{c}$ is the $n \times T$ matrix of observations on the n cash rates, and $\boldsymbol{\varepsilon}$ is the $n \times T$ matrix of T observations on the n currency appreciation rates, all with $T > 2n$. Consequently, $[\mathbf{r} - \mathbf{c}]$ is the $n \times T$ matrix of T observations on the n country risk premia and $[\mathbf{c} + \boldsymbol{\varepsilon}]$ is the $n \times T$ matrix of observations on the n country cash earning rates. Here

$$\mathbf{H} = \begin{bmatrix} [\boldsymbol{\iota}_n \otimes \mathbf{I}_n] & [\mathbf{I}_n \otimes \boldsymbol{\iota}_n] \end{bmatrix} \tag{44}$$

is the crucial $n^2 \times 2n$ selector matrix, which embodies the exact cash accounting identities, *i.e.*, the exact cash accounting framework.

4.1 Exact Investment Risk Attribution

In this section vectorization is implemented to prepare for the mean-variance analysis. The holding horizon averages of each of the n strategies are given by the $n^2 \times 1$ vector

$$\overline{\mathbf{VEC(S)}} = \frac{\mathbf{VEC(S)}.\boldsymbol{\iota}_T}{T} \tag{45}$$

where $\boldsymbol{\iota}_T$ is the $T \times 1$ unit vector. Thus

$$\overline{\mathbf{VEC(S)}} = \mathbf{H} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ \\ [\overline{\mathbf{c}} + \overline{\varepsilon}] \end{bmatrix} \tag{46}$$

We can now compute the strategy risk matrix Σ of the n^2 investment strategies and exactly decompose it into its various stock market and cash return risks

$$\begin{aligned} \boldsymbol{\Sigma} &= \frac{\mathbf{DEV(S).DEV(S)}'}{T} \\ &= \frac{\left\{ \mathbf{H} \begin{bmatrix} [\mathbf{r} - \mathbf{c}] \\ \\ [\mathbf{c} + \varepsilon] \end{bmatrix} [\mathbf{I}_T - \frac{\boldsymbol{\iota}_T \boldsymbol{\iota}_T'}{T}] \right\} \left\{ \mathbf{H} \begin{bmatrix} [\mathbf{r} - \mathbf{c}] \\ \\ [\mathbf{c} + \varepsilon] \end{bmatrix} [\mathbf{I}_T - \frac{\boldsymbol{\iota}_T \boldsymbol{\iota}_T'}{T}] \right\}'}{T} \\ &= \mathbf{H\Phi H}' \end{aligned} \tag{47}$$

where $\boldsymbol{\Phi}$ is the $2n \times 2n$ risk premium - cash return covariance matrix.

4.2 Mean-Variance Portfolio Optimization

For strategic global portfolio management the fundamental question is: are there combinations of investment strategies that either lead to lower overall risk for a comparable level of average return, or, *vice versa*, to a higher return for a comparable level of risk? Markowitz' original mean-variance optimization of portfolios, which answers this question, requires that the central risk matrix is positive definite, *i.e.*, nonsingular (Markowitz, 1952, 1991). Markowitz reformulated his problem in the 1980s to incorporate the possibility of a positive semi-definite risk matrix

$\mathbf{\Sigma}$.[14] But since the strategy risk matrix $\mathbf{\Sigma}$ based on exact accounting must be singular and we want to adopt Markowitz' optimization, not by fudging or avoiding, but by exploiting the exact accounting identities, the following simple exposition was developed using simple tensor algebra and the Kuhn-Tucker Theorem.

4.2.1 Singularity of Strategy Risk Matrix

It is easy to proof that the strategy risk matrix is singular, but this fact has stymied a lot of portfolio managers. It must singular, since

$$rank(\mathbf{\Sigma}) = 2n - 1 < n^2, \text{ for integer } n > 1 \tag{48}$$

This is proved by determining the rank of the strategy deviations $\mathbf{DEV}(\mathbf{S})$, which is the same as the rank of the strategy risk matrix $\mathbf{\Sigma}$.

$$\begin{aligned} rank\left\{\mathbf{DEV}(\mathbf{S})\right\} &= rank\left\{\mathbf{H}\begin{bmatrix} [\mathbf{r}-\mathbf{c}] \\ [\mathbf{c}+\varepsilon] \end{bmatrix}[\mathbf{I}_T - \frac{\boldsymbol{\iota}_T\boldsymbol{\iota}_T'}{T}]\right\} \\ &\leq Min\left\{rank\left(\mathbf{H}\right), rank\begin{bmatrix} [\mathbf{r}-\mathbf{c}] \\ [\mathbf{c}+\varepsilon] \end{bmatrix}, rank\left[\mathbf{I}_T - \frac{\boldsymbol{\iota}_T\boldsymbol{\iota}_T'}{T}\right]\right\} \\ &= Min\left\{2n-1, 2n, T-1\right\} = 2n - 1 < n^2 \end{aligned} \tag{49}$$

This result can be illustrated by computing the rank of the selector matrix $\mathbf{H}$ for our nine countries. Since $\mathbf{H}$ consists of zeros and ones only, this is easily done. For our empirical example

$$rank(\mathbf{H}) = 2n - 1 < n^2 \tag{50}$$

for $N > 2$.

[14] Since Markowitz (1987) is currently out of print, Professor Markowitz drew my attention to the generality of his 1980s presentation in reaction to Los (1998), by courteously sending me a copy of his book. Of course, one may question the relevance of symmetric mean-variance optimization, since there are observable asymmetries in the regional risk distributions. An optimization based on the empirical asymmetrical and leptokurtic return distributions would require the computation of third and fourth moments. But then the procedure of this would quickly become quickly very complex, without elucidating the issues of combining portfolio optimization with complete and exact attribution and the resulting singularity of the risk matrix.

4.2.2 Extended Markowitz Procedure

How we can exploit the accounting identities to get around the problem of the singular strategy risk matrix? The mean portfolio rate of return $\overline{s}_p$ for the holding period T is the allocated linear combination of the average strategy rates of return of the strategies, $\overline{\mathbf{VEC(S)}}$, contained in the portfolio

$$\begin{aligned} \overline{s}_p &= \mathbf{w}'\mathbf{VEC}(\mathbf{S}) \\ &= \mathbf{w}'\mathbf{H}\begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \\ &= \mathbf{v}'\begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \end{aligned} \tag{51}$$

where w is a $n^2 \times 1$ vector of portfolio allocations, such that the sum of the allocations equals unity, $w'\iota_{n^2} = 1$, where ι_{n^2} is the $n^2 \times 1$ unit vector; v is the $2n \times 1$ vector of the combined portfolio allocations $v = H'w$, such that $v'\iota_{2n} = 2$, where ι_{2n} is the $2n \times 1$ unit vector. The overall investment strategy portfolio risk, σ_{pp}, is the variance of the portfolio rates of return,

$$\sigma_{pp} = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \mathbf{w}'\mathbf{H}\boldsymbol{\Phi}\mathbf{H}'\mathbf{w} = \mathbf{v}'\boldsymbol{\Phi}\mathbf{v} \tag{52}$$

Notice that the first n combined allocations $v_{i.}, i = 1, 2, ...n$, refer to the strategy choice of how much of the capital to invest in which stock market to earn a risk premium, while the second n allocations $v_{.j}, j = 1, 2, ..., n$, refer to the strategy choice of how much of the capital to invest in which currency to earn a cash return. The allocations $\mathbf{v}$ exhaust the capital allocation based on these two fundamental choices of investments in stock markets and in currencies, because of the accounting identities,

$$\begin{aligned}
\mathbf{v}' &= \mathbf{v}' \left[\begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} \right] \text{ with} \\
\mathbf{v}' \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} &= \mathbf{w}'\mathbf{H} \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} = \mathbf{w}'\boldsymbol{\iota}_{n^2} = 1 \text{ and} \\
\mathbf{v}' \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} &= \mathbf{w}'\mathbf{H} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} = \mathbf{w}'\boldsymbol{\iota}_{n^2} = 1, \qquad (53)
\end{aligned}$$

Now the procedure has once again become similar to Markowitz' original nonsingular case, which we solve using the familiar Kuhn-Tucker Theorem for constraint optimization. First, form the Lagrangian with the three accounting constraints:

$$L(\mathbf{v}, \lambda_1, \lambda_2, \lambda_3) = \mathbf{v}'\boldsymbol{\Phi}\mathbf{v} + \lambda_1 \left[1 - \mathbf{v}' \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} \right] + \lambda_2 \left[1 - \mathbf{v}' \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} \right] + \lambda_3 \left[\overline{s}_p - \mathbf{v}' \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \right] \qquad (54)$$

Next, to find the optimum of this Lagrangian, set the $2n + 3$ partial first derivatives equal to zero, *i.e.*, the derivatives with respect to the $2n$ elements of the allocation vector $\mathbf{v}$ and to the two Lagrangian multipliers λ_1, λ_2 and λ_3.

The optimal Lagrangian multipliers λ_1^{opt}, λ_2^{opt} and λ_3^{opt} are given by

$$\begin{bmatrix} \lambda_1^{opt} \\ \lambda_2^{opt} \\ \lambda_3^{opt} \end{bmatrix} = 2.\boldsymbol{\Delta}^{-1}. \begin{bmatrix} 1 \\ 1 \\ \overline{s}_p \end{bmatrix} \qquad (55)$$

where the 3×3 symmetric and positive definite matrix $\mathbf{\Delta}$ is such that

$$\mathbf{\Delta} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{\iota}_n' & \mathbf{0} \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{0} & \boldsymbol{\iota}_n' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}]' & [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}]' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{\iota}_n' & \mathbf{0} \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} & \begin{bmatrix} \mathbf{0} & \boldsymbol{\iota}_n' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} & \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}]' & [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}]' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{\iota}_n' & \mathbf{0} \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} & \begin{bmatrix} \mathbf{0} & \boldsymbol{\iota}_n' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} & \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}]' & [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}]' \end{bmatrix} \mathbf{\Phi}^{-1} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \end{bmatrix} \tag{56}$$

So that by substitution, the optimal fundamental choice allocations are determined to be

$$\mathbf{v}^{opt} = \frac{\mathbf{\Phi}^{-1} \left[\lambda_1^{opt} \begin{bmatrix} \boldsymbol{\iota}_n \\ \mathbf{0} \end{bmatrix} + \lambda_2^{opt} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\iota}_n \end{bmatrix} + \lambda_3^{opt} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \right]}{2} \tag{57}$$

The $(2n+3) \times (2n+3)$ matrix of partial second derivatives is positive definite, since the full rank $2n \times 2n$ covariance matrix $\mathbf{\Phi} > 0$, so that the optimum is, indeed, a - constrained - *minimum.*

We have now the two equations of Markowitz' Efficient Portfolio Frontier for $2n$ strategy investment choices, which can be plotted in a two-dimensional graph. For every portfolio strategy return $\overline{s}_p^{opt}$ there is a corresponding portfolio strategy risk σ_p^{opt}:

$$\overline{s}_p^{opt} = \mathbf{v}^{opt\prime} \begin{bmatrix} [\overline{\mathbf{r}} - \overline{\mathbf{c}}] \\ [\overline{\mathbf{c}} + \overline{\boldsymbol{\varepsilon}}] \end{bmatrix} \tag{58}$$

$$\sigma_p^{opt} = \sqrt{(\mathbf{v}^{opt})\mathbf{\Phi}'\mathbf{v}^{opt}} \tag{59}$$

4.3 Retrieval of the Optimal Strategy Allocations

The original portfolio investment strategy portfolio allocations w can be uniquely retrieved from the fundamental market choice portfolio allocations v, via the two accounting identities imposed by the geometry of the original strategy matrix $\mathbf{S}(t)$, for example, when $n = 3$,

$$\begin{aligned}
\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{v}\mathbf{v}' \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} &= \begin{bmatrix} v_1 v_4 & v_1 v_5 & v_1 v_6 \\ v_2 v_4 & v_2 v_5 & v_2 v_6 \\ v_3 v_4 & v_3 v_5 & v_3 v_6 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{H}'\mathbf{w}\mathbf{w}'\mathbf{H} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_n \end{bmatrix} \\
&= \mathbf{W} \\
&= \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}
\end{aligned} \tag{60}$$

Thus the investment strategy portfolio allocations $\mathbf{w}$ are simply the *products* of the fundamental market choice portfolio allocations.

4.4 Empirical Example: Asian Multi-currency Portfolio Optimization

Fig. 5. provides the efficiency frontier of the strategic multi-currency investment portfolios with exact risk attribution for the period July 1992 - December 1997 for $n = 9$ countries - 7 Asian + Germany + USA -, given by the bottom parabola. We compare this with the parabolic efficiency frontiers of the 7 Asian countries by themselves, given by the top parabola, and with the parabolic efficiency frontiers, when either Germany or the USA alone is included in the Asian portfolio. The optimal portfolio risk σ_p^{opt} is measured on the vertical axis, while the average portfolio return is measured on the horizontal axis.

[PLACE FIG. 5 ABOUT HERE]

Table 5 provides the optimal portfolio allocations for all nine countries combined based on the total investment of a five and a half year investment horizon (July 1992 - December 1997) These are the allocations corresponding with the General Minimum-Variance (GMV) allocation, *i.e.*, for the absolute minimum risk point of the lowest parabolic efficiency frontier in Figure 1, where the average optimal portfolio return is $\overline{s}_p^{opt} = 13.7\%$/yr and the average portfolio risk $\sigma_p^{opt}(GMV) = 25.3\%$ within one year. Along the green left column one can read the allocation weights for the fundamental risk premiums, while along the orange top row one can read the allocation weights for the the cash overlay returns. The yellow cells in the middle represent the combined (product) capital allocation weights.

[PLACE TABLE 5 ABOUT HERE]

5 CONCLUSIONS

The field of Finance is rapidly evolving and in the process it borrows concepts from different disciplines. Not all these concepts are properly applied, or they are only applied in a very limited, and often historically predetermined way. In this paper we discuss three such important borrowed modeling concepts: uncertainty, complexity and rank. For simplicity, the discussion of these concepts is presented in the static context of stationary processes, but can be extended to non-stationary data, with time-varying first and second moments, which can be filtered by the usual Kalman filter. Furthermore, the discussion is restricted to financial systems which are linear in the Grassmanian coefficients. For financial applications, we look at the CAPM, the Dividend Discount Model, the Asset-Pricing Model, and Mean-Variance Portfolio Optimization with Exact Attribution.

First, it was noticed that modeling uncertainty is much a much broader phenomenon than the unidirectional projection residuals measured in "regression analysis", which are often presumed to adhere to a presumed probability distribution, often Gaussian. This statistical presumption is then the basis for the statistical game of "significance "or "confidence "measurement and "hypothesis testing. " But measurement uncertainty is a multi-dimensional phenomenon and therefore should be measured in all directions of the observed n-dimensional data input set by a complete set of projections. Unidirectional testing based on a presumed statistical characterization of the residual noise can thus be extremely misleading! This paper proposes a simple, non-statistical, multi-dimensional Noise/Data ratio measure, which can always be computed. This discussion is illustrated by a simple application of Complete Least Squares (CLS) projections to the empirical computation of the relative risk, CAPM betas of American mutual funds. The conclusion from that illustration is that the usual categorization into aggressive, neutral and defensive investments must be supplemented by the category "undecided" to be complete. Most so-called defensive assets appear to be actually "undecided," since the data uncertainty does not allow a definite categorization of the "beta" or relative market risk.

Second, the discussion of the concept of modeling complexity forces one to accept that most published financial and economic research of the past five decades is incomplete. It ignores most of the linear system information available in the $\frac{n(n-1)}{2}$ independent bivariate covariances of the data covariance matrix $\mathbf{\Sigma}$, or more precisely, in the n rows of the information matrix $\mathbf{\Sigma}^{-1}$, where each row is an elementary $(n,q) = (n,1)$ ("regression") projection. The paper suggests that the complete search for systematic relationships is much more complex than financial and economic researchers usually assume. In particular, the presumption of single equation systems for $n > 2$ variables is almost always falsified by the complete $\mathbf{\Sigma}^{-1}$. For small and moderate noise levels, the corank q of the system must can often be inferred from the $\mathbf{\Sigma}^{-1}$ by careful examination of the complete set of (n,q) LS projections, where $1 < q < n$.

This complete set of LS projections is easily generated by the new Complete Least Squares (CLS) estimator. The complete set includes all basic orthogonal data projections. This second discussion is illustrated by an application to the identification of simple systems between the stock market indices, Real GDP and short term interest rates of six Asian countries, in particular of Taiwan. It is concluded that one country exhibits a $(n,q) = (3,2)$ system, with two simultaneous equations; four countries exhibit simple $(n,q) = (2,1)$ bivariate relationships, *i.e.* they exhibit neither $(n,q) = (3,2)$ nor $(n,q) = (3,1)$ systems among the three variables; and one country's data are so uncertain that no systematic (linear-log) relationship can be established at all. Considering that many global portfolio managers and country risk analysts automatically assume the existence of, often very simplistic, systems, this conclusion should give pause for more thought.

Third, the discussion of modeling addresses a numerical instability problem often encountered by professional fund managers, who try to optimize their portfolios with standardized, off-the-shelf portfolio optimizing software. It focuses on the rank conditions imposed on portfolio management by exact accounting identities of discounted cash flows of fundamental and contingent assets and liabilities (derivatives). Often this rank condition is overlooked by professional portfolio managers and by the software packages they implement for the optimization of their portfolios, although the problem it causes was already acknowledged by Markowitz in 1987. When properly recognized, the portfolio optimization problem can be resolved precisely by applying proper tensor algebra to the exact risk attribution problem, and the numerical instability vanishes when the smaller full rank covariance matrix is inverted. This is illustrated by an implementation to a multi-currency, multi-country stock market portfolio of nine countries, seven Asian countries plus the USA and Germany.

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7 TABLES

TABLE 1. SCIENTIFIC INCOMPLETENESS			
Article (Type of Analysis)	**# of**	**Analytic**	**# of unreported**
	Variables	**Incompleteness %**	**LS projections**
Fama (1990) (LS)	5	80	29
Schwert (1990) (LS)	5	80	29
	9	89	509
	12	92	4,093
Bittlingmayer (1992) (LS)	8	88	253
Canova & De Nicolo (1995) (LS)	4	75	13
	6	83	61
	9	89	509
	11	91	2,045
Lee (1992) (VAR)	28	96	268,000,000
Gallinger (1994) (ADF)	13	92	8,189
	15	93	32,765
	17	94	131,060
	18	94	262,141
	19	95	542,285

	Table 2. Systematic Risk of Mutual Funds	#	%
1.	Morningstar's Principia for Mutual Funds universe, 12/31/95	7,051	
2.	Together with the condition $0 < \rho_{12}^2 \leq 1$	3,227	
3.	And with 3-year (Sharpe's) beta $0 < \beta_2$	3,215	
4.	**AIMR Performance Presentation Standards, 1993, 1996:**		
	(i) Defensive funds: $0 < \beta_2 < 1$	2,047	63.7
	(ii) Neutral, market index funds: $\beta_2 = 1$	67	2.1
	(iii) Aggressive funds: $1 < \beta_2$	1,101	34.2
	Total funds with measurable systematic market risk	3,215	100.0
5.	**CLS analysis:**		
	(i) Defensive funds: $0 < \beta_2 \leq \beta_1 < 1$	608	18.9
	(ii) Neutral, market index funds: $\beta_2 = \beta_1 = 1$	18	0.6
	(iii) Aggressive funds: $1 < \beta_2 \leq \beta_1$	1,101	34.2
	(iv) Undecided: $0 < \beta_2 < 1 < \beta_1$	1,488	46.3
	Total funds with measurable systematic market risk	3,215	100.0

TABLE 3.	INCOME (c) AND	INTEREST RATE (d)	ELASTICITIES
From ($q = 2$) CLS Projections	$(c, d)_1$	$(c, d)_2$	$(c, d)_3$
TAIWAN	$+3.623, +2.817$	$+1.375, +2.145$	$+2.751, +1.084$
MALAYSIA	$+1.706, -9.433$	$+1.495, -13.1578$	$+3.541, -1.392$
SINGAPORE	$+1.205, -10.000$	$+1.030, -12.561$	$+1.513, -0.959$
PHILIPPINES	$+2.012, -12.500$	$+1.612, -537.333$	$+104.560, -0.763$
INDONESIA	$+13.210, -21.505$	$+7.632, -19.384$	$+11.902, -1.731$
JAPAN	$-13.158, +3.676$	$-0.278, +1.188$	$-4.253, +0.040$

TABLE 4.	IDENTIFICATION OF SYSTEM INVARIANT CORANK q				
From:	**Spectral Analysis of Σ**	**Information Matrix Σ^{-1}**	**Inspection of (a,b) Plots**	**3D CLS Plots**	**3D N/D Ratio**
TAIWAN	$r = 1, q = 2$	2	2	**2**	0.37
MALAYSIA	$r = 1; q = 2$	1	1 ($\mathbf{x}_3$ = noise)	**1**	0.07
SINGAPORE	$r = 1; q = 2$	1	1 ($\mathbf{x}_3$ = noise)	**1**	0.13
PHILIPPINES	$r = 1; q = 2$	1	1 ($\mathbf{x}_3$ = noise)	**1**	0.14
INDONESIA	$r = 1; q = 2$	2	2 ($\mathbf{x}_3$ = noise)	**1**	0.39
JAPAN	$r = 3; q = 0$	2	2 ($\forall \mathbf{x}_i$ = noise)	**0**	0.86

TABLE 5: OPTIMAL GMV STRATEGIC ALLOCATIONS FOR ASIAN COUNTRIES, TOGETHER WITH USA AND GERMANY

July 1992 - December 1997

Average portfolio return = 13.7%
Average portfolio risk (stdev) = 25.3%

			Hong Kong	Indonesia	Japan	Malaysia	Philippines	Singapore	Thailand	Germany	USA	
	cash return weights		$v_{.1}$	$v_{.2}$	$v_{.3}$	$v_{.4}$	$v_{.5}$	$v_{.6}$	$v_{.7}$	$v_{.8}$	$v_{.9}$	
	risk premium weights		110.5%	-48.7%	11.4%	41.4%	41.6%	-99.2%	11.8%	25.4%	5.9%	100.0%
Hong Kong	$v_{1.}$	-13.7%	-15.2%	6.7%	-1.6%	-5.7%	-5.7%	13.6%	-1.6%	-3.5%	-0.8%	-13.7%
Indonesia	$v_{2.}$	-14.1%	-15.6%	6.9%	-1.6%	-5.8%	-5.9%	14.0%	-1.7%	-3.6%	-0.8%	-14.1%
Japan	$v_{3.}$	6.2%	6.9%	-3.0%	0.7%	2.6%	2.6%	-6.2%	0.7%	1.6%	0.4%	6.2%
Malaysia	$v_{4.}$	11.4%	12.6%	-5.5%	1.3%	4.7%	4.7%	-11.3%	1.3%	2.9%	0.7%	11.4%
Philippines	$v_{5.}$	2.7%	3.0%	-1.3%	0.3%	1.1%	1.1%	-2.7%	0.3%	0.7%	0.2%	2.7%
Singapore	$v_{6.}$	11.6%	12.8%	-5.6%	1.3%	4.8%	4.8%	-11.5%	1.4%	2.9%	0.7%	11.6%
Thailand	$v_{7.}$	3.7%	4.1%	-1.8%	0.4%	1.5%	1.5%	-3.7%	0.4%	0.9%	0.2%	3.7%
Germany	$v_{8.}$	11.9%	13.2%	-5.8%	1.4%	4.9%	5.0%	-11.8%	1.4%	3.0%	0.7%	11.9%
USA	$v_{9.}$	80.3%	88.7%	-39.1%	9.1%	33.2%	33.4%	-79.7%	9.4%	20.4%	4.7%	80.3%
		100.0%	110.5%	-48.7%	11.4%	41.4%	41.6%	-99.2%	11.8%	25.4%	5.9%	200.0%

8 FIGURES

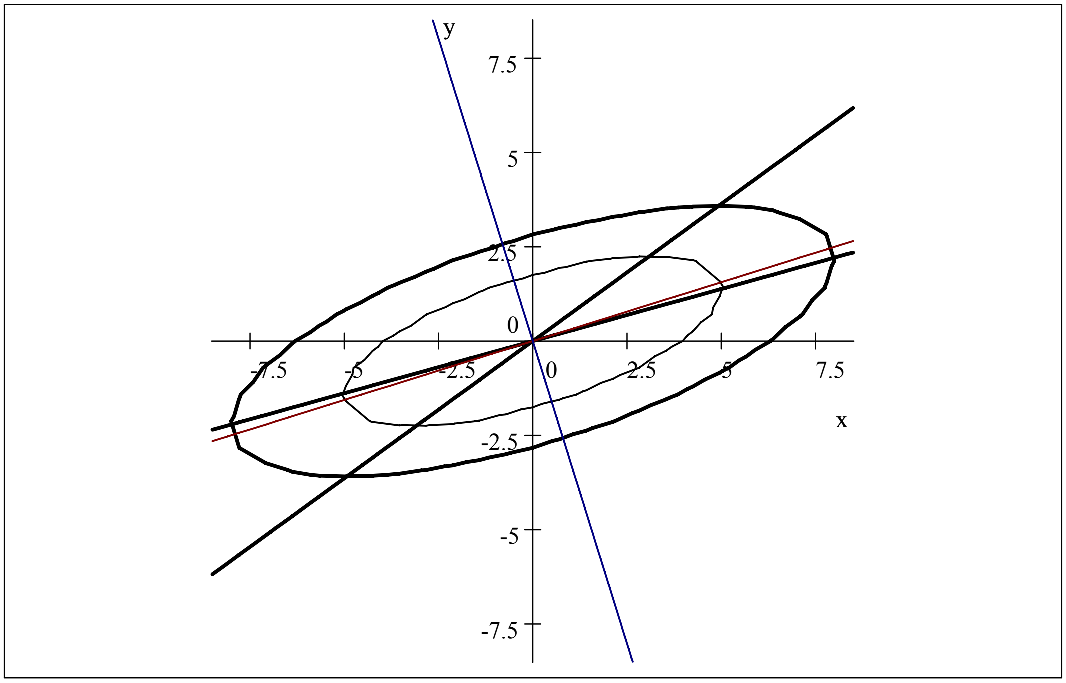

Fig. 1: Exact information ellipses and LS orthogonal projection lines in black. The principal axis is in red and the minor axis in blue.

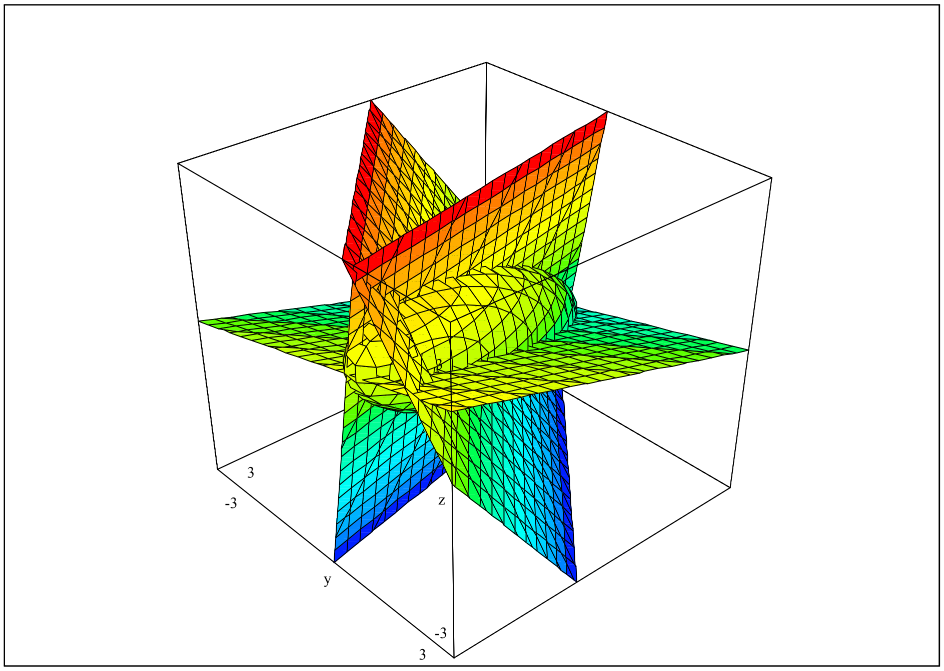

Fig. 2: Exact information ellipsoid and the three LS projection planes found by projection one of the three variables on the other two.

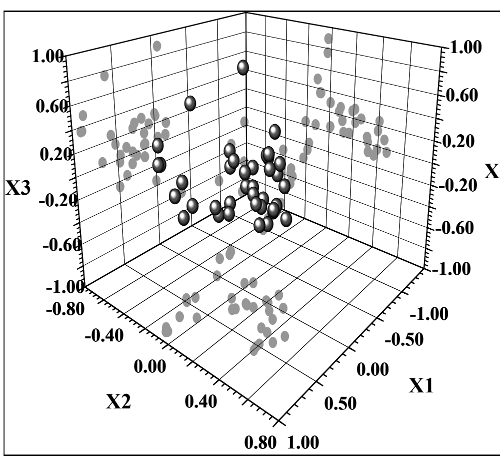

Fig. 3. Taiwan 3D Scatterplot

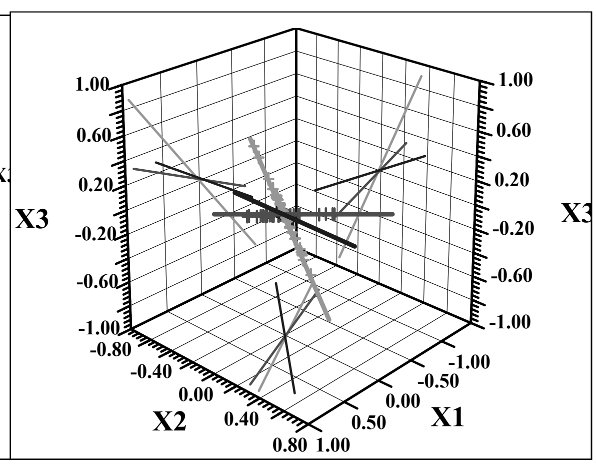

Fig. 4. Taiwan 3D CLS Plot

FIGURE 5: Efficient Portfolio Frontiers of Strategic Multi-Currency Investments in Asia, July 1992 - December 1997