Dynamic Factor Analysis with Nonlinear Temporal Aggregation Constraints

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Abstract

The paper estimates an index of coincident economic indicators for the U.S. economy using time series with different frequencies of observation (monthly and quarterly, possibly with missing values). The model considered is the dynamic factor model proposed by Stock and Watson, specified in the logarithms of the original variables and at the monthly frequency, which poses a problem of temporal aggregation with a nonlinear observational constraint when quarterly time series are included. Our main methodological contribution is to provide an exact solution to this problem, that hinges on conditional mode estimation by extended Kalman filtering and smoothing. On the empirical side the contribution of the paper is to provide monthly estimates of quarterly indicators, among which Gross Domestic Product, that are consistent with the quarterly totals.

Keywords: Nonlinearity. Disaggregation. State Space Models. Business Cycle.

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1 Introduction

A prominent feature of the business cycle is the presence of similarities in the dynamics of several representative series or, following Lucas (1977), *co-movements*. This notion is already attested in the classical business cycle definition due to Burns and Mitchell (1946), according to whom business cycle fluctuations "take place almost at the same time in many economic activities (. . .)". Hence, this feature implies that the reference cycle cannot be extracted from a single series, e.g. Gross Domestic Product (GDP), but it calls for the analysis of a range of relevant indicators of economic activity.

Stock and Watson (1991, SW henceforth) developed an explicit probability model for the composite index of coincident economic indicators. They proposed a dynamic factor model featuring a common difference-stationary factor that defines the composite index. The reference cycle is assumed to be the value of a single unobservable variable, "the state of the economy", that by assumption represents the only source of the co-movements of four time series: industrial production, sales, employment, and real incomes.

On the other hand, GDP is perhaps the most important coincident indicator, although it is available only quarterly and it is subject to greater revisions than the four coincident series in the original SW model. This consideration motivated Mariano and Murasawa (2003, MM hereafter) to extend the SW model with the inclusion of quarterly real GDP growth, proposing a linear state space model at the monthly observation frequency that entertains the presence of an aggregated flow. Although their model is formulated explicitly in terms of the logarithmic changes in the variables, the nonlinear nature of the temporal aggregation constraint is not taken into account.

This paper proposes several refinements to this literature: first and foremost, the problem of modelling time series with different frequencies of observations and subject to a nonlinear temporal aggregation constraint, induced by the logarithmic transformation, is explicitly afforded. The solution we propose is grounded in the theory developed by Fahrmeir (1992) and Durbin and Koopman (2001), and requires matching the conditional mode of the states of the nonlinear and the linear approximation; this operation is performed by iterating on the Kalman filter and smoother estimating equations.

Secondly, the model is set up in the log-levels of the variables rather than in the changes of their logarithms. The advantages of this formulation are twofold: in the first place the mean square error of the estimated coincident index are immediately available both in real time (filtering) and after processing the full available sample (smoothing). Moreover, the treatment of the aggregation constraint in the log-levels is more transparent and efficient from the computational standpoint, in that it leads to a reduced state vector dimension.

The paper is organized as follows: Section 2 introduces the level formulation of the original SW coincident index model, and Section 3 casts the model in the linear state space form. Section 4 discusses how the latter is modified in order to account for the the presence of temporally aggregated flow variables. The nonlinear temporal aggregation constraint that arises when the series are modelled in their logs is dealt with in Section 5, where we discuss inference on the unobserved components using the technique of posterior mode estimation and maximum likelihood estimation. The empirical illustration, presented in Section 6, refers to the estimation of an index of coincident indicators for the U.S. economy. Section 7 draws some conclusions.

2 The level specification of the index model

The coincident index model proposed by SW, aims at rationalizing by a probabilistic model the judgmental procedure used by the Department of Commerce to build up a coincident indicator for the U.S. economy. The fundamental idea is to separate the dynamics which are common to a set of N coincident series, $\mathbf{y}_t$, that are $I(1)$ but not cointegrated, from the idiosyncratic component, which is specific to each series.

The level specification of the SW *single index* model expresses $\mathbf{y}_t$, possibly after a logarithmic transformation, as the linear combination of a common cyclical trend, that will be denoted by μ_t, and an idiosyncratic component, $\boldsymbol{\mu}_t^*$. Letting $\boldsymbol{\theta}$ denote an $N \times 1$ vector of loadings, and assuming that both components are difference stationary and subject to autoregressive dynamics, we can write:

$$\begin{array}{rcll}\mathbf{y}_t & = & \boldsymbol{\theta}\mu_t + \boldsymbol{\mu}_t^*, & t = 1, ..., n, \\ \phi(L)\Delta\mu_t & = & \eta_t, & \eta_t \sim \text{NID}(0, \sigma_\eta^2), \\ \mathbf{D}(L)\Delta\boldsymbol{\mu}_t^* & = & \boldsymbol{\beta} + \boldsymbol{\eta}_t^*, & \boldsymbol{\eta}_t^* \sim \text{NID}(\mathbf{0}, \boldsymbol{\Sigma}_{\eta^*}),\end{array} \tag{1}$$

where $\phi(L)$ is an autoregressive polynomial of order p with stationary roots:

$$\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$$

and the matrix polynomial $\mathbf{D}(L)$ is diagonal:

$$\mathbf{D}(L) = \text{diag}\left[d_1(L), d_2(L), \ldots, d_N(L)\right],$$

with $d_i(L) = 1 - d_{i1}L - \cdots - d_{ip_i}L^{p_i}$ and $\boldsymbol{\Sigma}_{\eta^*} = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2)$. The disturbances η_t and $\boldsymbol{\eta}_t^*$ are mutually uncorrelated at all leads and lags.

The state vector features $N + 1$ additional elements with respect to the original SW formulation based on $\Delta\mathbf{y}_t$. However, the representation (1) eliminates the ambiguities in the interpretation of the real time (filtered) and smoothed estimates that arise when the model is formulated in terms of differences; for an account see also MM. Notice that (1) assumes a zero drift for the single index. Moreover, the level representation is also more amenable for the treatment of temporal aggregation.

Note that both μ_t and $\boldsymbol{\mu}_t^*$ are difference stationary processes and the common dynamics are the results of the accumulation of the same underlying shock η_t; moreover, the process generating the index of coincident indicators is usually more persistent than a random walk and in the accumulation of the shocks produces cyclical swings.

3 State space representation

In this section we cast model (1) in the state space form (SSF). We start from the single index, $\phi(L)\Delta\mu_t = \eta_t$, considering the SSF of the stationary AR(p) model for the $\Delta\mu_t$, for which:

$$\begin{aligned}\Delta\mu_t &= \mathbf{e}'_{1p}\mathbf{a}_t,\\ \mathbf{a}_t &= \mathbf{T}_{\Delta\mu}\mathbf{a}_{t-1} + \mathbf{e}_{1p}\eta_t,\end{aligned}$$

where $\mathbf{e}_{1p} = [1, 0, \ldots, 0]'$ and

$$\mathbf{T}_{\Delta\mu} = \begin{bmatrix} \phi_1 & \\ \vdots & \mathbf{I}_{p-1} \\ \phi_{p-1} & \\ \phi_p & \mathbf{0}' \end{bmatrix}.$$

Hence, $\mu_t = \mu_{t-1} + \mathbf{e}'_{1p}\mathbf{a}_t = \mu_{t-1} + \mathbf{e}'_{1p}\mathbf{T}_{\Delta\mu}\mathbf{a}_{t-1} + \eta_t$, and defining

$$\boldsymbol{\alpha}_{\mu,t} = \begin{bmatrix} \mu_t \\ \mathbf{a}_t \end{bmatrix}, \quad \mathbf{T}_\mu = \begin{bmatrix} 1 & \mathbf{e}'_{1p}\mathbf{T}_{\Delta\mu} \\ 0 & \mathbf{T}_{\Delta\mu} \end{bmatrix},$$

the Markovian representation of the model for μ_t becomes

$$\mu_t = \mathbf{e}'_{1,p+1}\boldsymbol{\alpha}_{\mu,t}, \quad \boldsymbol{\alpha}_{\mu,t} = \mathbf{T}_\mu\boldsymbol{\alpha}_{\mu,t-1} + \mathbf{R}_\mu\eta_t,$$

where $\mathbf{R}_\mu = [1, \mathbf{e}'_{1,p}]'$.

A similar representation holds for each individual μ^*_{it}, with ϕ_j replaced by d_{ij}, so that, if we let p_i denote the order of the i-th lag polynomial $d_i(L)$, we can write:

$$\mu^*_{it} = \mathbf{e}'_{1,p_i+1}\boldsymbol{\alpha}_{\mu_i,t}, \quad \boldsymbol{\alpha}_{\mu_i,t} = \mathbf{T}_i\boldsymbol{\alpha}_{\mu_i,t-1} + \mathbf{c}_i + \mathbf{R}_i\eta^*_{it},$$

where $\mathbf{R}_i = [1, \mathbf{e}'_{1,p_i}]'$, $\mathbf{c}_i = \beta_i\mathbf{R}_i$ and β_i is the drift of the $i-th$ idiosyncratic component, and thus of the series, since we have assumed a zero drift for the common factor.

Combining all the blocks, we obtain the SSF of the complete model by defining the state vector $\boldsymbol{\alpha}_t$, with dimension $\sum_i (p_i+1) + p + 1$, as follows:

$$\boldsymbol{\alpha}_t = [\boldsymbol{\alpha}'_{\mu,t}, \boldsymbol{\alpha}'_{\mu_1,t}, \ldots, \boldsymbol{\alpha}'_{\mu_N,t}]'. \tag{2}$$

Consequently, the measurement and the transition equation of SW model in levels is:

$$\mathbf{y}_t = \mathbf{Z}\boldsymbol{\alpha}_t, \quad \boldsymbol{\alpha}_t = \mathbf{T}\boldsymbol{\alpha}_{t-1} + \mathbf{c} + \mathbf{R}\boldsymbol{\epsilon}_t, \tag{3}$$

where $\boldsymbol{\epsilon}_t = [\eta_t, \eta^*_{1t}, \ldots, \eta^*_{Nt}]'$ and the system matrices are given below:

$$\begin{aligned} \mathbf{Z} &= \left[\boldsymbol{\theta} \vdots \ \mathrm{diag}(\mathbf{e}'_{p_1}, \ldots, \mathbf{e}'_{p_N})\right], & \mathbf{T} &= \mathrm{diag}(\mathbf{T}_\mu, \mathbf{T}_1, \ldots, \mathbf{T}_N), \\ \mathbf{c} &= [\mathbf{0}', \mathbf{c}'_1, \ldots, \mathbf{c}'_N]', & \mathbf{R} &= \mathrm{diag}(\mathbf{R}_\mu, \mathbf{R}_1, \ldots, \mathbf{R}_N). \end{aligned} \tag{4}$$

4 Temporal aggregation

In practical applications the coincident indicators may be observed at different frequencies, as it occurs in the U.S. case, for which GDP is quarterly, whereas retail sales, employment and industrial production are monthly.

In dealing with time series observed at different frequencies we need to operate a distinction between flows and stocks variables. For the former the aggregated series arises from the cumulative sum of the disaggregated measures over a larger time interval, and the problem is that of distributing the aggregate on shorter intervals. For the latter, the series observed at a lower frequency may arise as a systematic sample of the disaggregated one, in which case estimation at points between observations is termed ”interpolation”; on the other hand, if that series is obtained by taking the time average of the disaggregated stock, the situation is the same as for flows. Since in the sequel we shall deal only with flow variables and time-averaged stocks, our discussion will be restricted to this particular type of temporal aggregation.

The approach to the treatment of mixed frequency series that we adopt is that proposed by Harvey (1989, sec. 6.3), who considered it as a problem of missing observations in the aggregated time series (see also Harvey and Pierse, 1984), within a suitably modified representation of the model. Suppose that the set of coincident indicators, $\mathbf{y}_t$, can be partitioned into two groups, $\mathbf{y}_t = [\mathbf{y}'_{1t}, \mathbf{y}'_{2t}]'$, where the second block gathers the flows or time averaged stocks that are subject to temporal aggregation, so that

$$\mathbf{y}^*_{2\tau} = \sum_{i=0}^{\delta-1} \mathbf{y}_{2,\tau\delta-i}, \quad \tau = 1, 2, \ldots, [T/\delta],$$

where δ denote the aggregation interval: for instance, if the model is specified at the monthly frequency and $\mathbf{y}^\dagger_{2t}$ is quarterly, then $\delta = 3$.

The strategy proposed by Harvey (1989) consists of operating a suitable augmentation of the state vector (2) using an appropriately defined cumulator variable. In particular, the SSF (3)-(6) need to be augmented by the $N_2 \times 1$ vector $\mathbf{y}^{\dagger}_{2t}$, generated as follows

$$\begin{aligned} \mathbf{y}^{\dagger}_{2t} &= \psi_t \mathbf{y}^{\dagger}_{2,t-1} + \mathbf{y}_{2t} \\ &= \psi_t \mathbf{y}^{\dagger}_{2,t-1} + \mathbf{Z}_2 \mathbf{T} \boldsymbol{\alpha}_{t-1} + \mathbf{Z}_2 \mathbf{c} + \mathbf{Z}_2 \mathbf{R} \boldsymbol{\epsilon}_t \end{aligned}$$

where ψ_t is the cumulator variable, defined as follows:

$$\psi_t = \begin{cases} 0 & t = \delta(\tau - 1) + 1, \quad \tau = 1, \dots, [T/\delta] \\ 1 & \text{otherwise} . \end{cases}$$

and $\mathbf{Z}_2$ is the $N_2 \times m$ block of the measurement matrix $\mathbf{Z}$ corresponding to the second set of variables, $\mathbf{Z} = [\mathbf{Z}_1', \quad \mathbf{Z}_2']'$ and $\mathbf{y}_{2t} = \mathbf{Z}_2 \boldsymbol{\alpha}_t$. Notice that at times $t = \delta\tau$ the cumulator coincides with the (observed) aggregated series, otherwise it contains the partial cumulative value of the aggregate in the seasons (e.g. months) making up the larger interval (e.g. quarter) up to and including the current one.

The augmented SSF is defined in terms of the new state and observation vectors:

$$\boldsymbol{\alpha}^*_t = \begin{bmatrix} \boldsymbol{\alpha}_t \\ \mathbf{y}^{\dagger}_{2t} \end{bmatrix}, \quad \mathbf{y}^{\dagger}_t = \begin{bmatrix} \mathbf{y}_{1t} \\ \mathbf{y}^{\dagger}_{2t} \end{bmatrix}$$

where the former has dimension $m^* = m + N_2$, and the unavailable second block of observations, $\mathbf{y}_{2t}$, is replaced by $\mathbf{y}^{\dagger}_{2t}$, which is observed at times $t = \delta\tau, \tau = 1, 2, \dots, [T/\delta]$, and is missing at intermediate times. The measurement and transition equation are therefore:

$$\mathbf{y}^{\dagger}_t = \mathbf{Z}^* \boldsymbol{\alpha}^*_t, \quad \boldsymbol{\alpha}^*_t = \mathbf{T}^* \boldsymbol{\alpha}^*_{t-1} + \mathbf{c}^* + \mathbf{R}^* \boldsymbol{\epsilon}_t, \tag{5}$$

with system matrices:

$$\mathbf{Z}^* = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{N_2} \end{bmatrix}, \quad \mathbf{T}^* = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{Z}_2 \mathbf{T} & \psi_t \mathbf{I} \end{bmatrix}, \quad \mathbf{c}^* = \begin{bmatrix} \mathbf{I} \\ \mathbf{Z}_2 \end{bmatrix} \mathbf{c}, \quad \mathbf{R}^* = \begin{bmatrix} \mathbf{I} \\ \mathbf{Z}_2 \end{bmatrix} \mathbf{R}. \tag{6}$$

The state space model (5)-(6) is linear and, assuming that the disturbances have a Gaussian distribution, the unknown parameters can be estimated by maximum likelihood, using the prediction error decomposition, performed by the Kalman filter; given the parameter values, the Kalman filter and smoother will provide the minimum mean square estimates of the states $\boldsymbol{\alpha}_t^*$ (see Harvey, 1989, and Shumway and Stoffer, 2000) and thus of the missing observations on $\mathbf{y}_{2t}^{\dagger}$ can be estimated, which need to be "decumulated", using $\mathbf{y}_{2t} = \mathbf{y}_{2t}^{\dagger} - \psi_t \mathbf{y}_{2,t-1}^{\dagger}$, so as to be converted into estimates of $\mathbf{y}_{2t}$.

5 Nonlinear temporal aggregation

Let us consider now the situation when $\mathbf{y}_t$ represents the logarithms of the original time series and the second block of series is temporally aggregated. This setting is more realistic, as $\Delta\mu_t$ captures the common component in the rate of change, rather than in the change itself, of the selected economic indicators.

The aggregation constraints is linear in $\mathbf{Y}_{2t} = \exp(\mathbf{y}_{2t})$, since the aggregated series results as follows:

$$\mathbf{Y}_{2\tau}^* = \sum_{i=0}^{\delta-1} \mathbf{Y}_{2,\tau\delta-i}. \tag{7}$$

The linear SSF of the previous section is no longer adequate, and yields distributed values that fail to satisfy the true aggregation constraint, i.e. the monthly value would not sum up (or average, in the case of time-averaged stocks) to quarterly totals.

Since the temporal aggregation constraint is nonlinear in $\mathbf{y}_t$, the resulting state space model is nonlinear. In the sequel we provide a theory of estimation and signal extraction for this model. The key to the results is approximate conditional mode estimation by extended Kalman filtering and smoothing, based on Durbin and Koopman (1992, 2001) and Fahrmeir (1992).

In order to derive the nonlinear SSF arising from model (3)-(4) under the nonlinear temporal aggregation constraint (7), we introduce a new cumulator variable, defined recursively as follows:

$$\begin{aligned} \mathbf{Y}_{2t}^{\dagger} &= \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp(\mathbf{y}_{2t}) \\ &= \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp\left(\mathbf{Z}_2 \boldsymbol{\alpha}_t\right). \end{aligned}$$

As in the previous case we augment the state vector by $\mathbf{Y}_{2t}^{\dagger}$, which however depends nonlinearly on $\boldsymbol{\alpha}_t$.

Given an arbitrary trial value $\tilde{\boldsymbol{\alpha}}_t$, the linear and Gaussian approximating model (LGAM) is obtained from the first order Taylor expansion of the cumulator around this value:

$$\begin{aligned}
\mathbf{Y}_{2t}^{\dagger} &= \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp(\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t) + \tilde{\mathbf{D}}_t\mathbf{Z}_2(\boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_t) \\
&= \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp(\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t) - \tilde{\mathbf{D}}_t\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t + \tilde{\mathbf{D}}_t\mathbf{Z}_2\mathbf{T}\boldsymbol{\alpha}_{t-1} + \tilde{\mathbf{D}}_t\mathbf{Z}_2\mathbf{c} + \tilde{\mathbf{D}}_t\mathbf{Z}_2\mathbf{R}\boldsymbol{\epsilon}_t
\end{aligned}$$

where $\tilde{\mathbf{D}}_t = \text{diag}(\mathbf{z}_{2i}'\tilde{\boldsymbol{\alpha}}_t)$, $\mathbf{z}_{2i}'$ denotes the i-th row of $\mathbf{Z}_2$, and we have replaced $\boldsymbol{\alpha}_t$ by the right hand side of the transition equation (3). In particular, $\tilde{\mathbf{D}}_t\mathbf{Z}_2$ is the matrix whose j-th row contain the derivatives of the j-th cumulator $Y_{jt}^{\dagger}$ with respect to $\boldsymbol{\alpha}_t'$, evaluated at the trial value $\tilde{\boldsymbol{\alpha}}_t$.

The SSF of the LGAM is based upon the augmented vector $\boldsymbol{\alpha}_t^{\dagger} = [\boldsymbol{\alpha}_t', \mathbf{Y}_{2t}^{\dagger'}]'$, with the measurement equation given by

$$\begin{bmatrix} \mathbf{y}_{1t} \\ \mathbf{Y}_{2t}^{\dagger} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \boldsymbol{\alpha}_t^{\dagger}, \tag{8}$$

where the left hand side lower block is observed only at $t = \tau\delta$, and transition equation:

$$\boldsymbol{\alpha}_t^{\dagger} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \tilde{\mathbf{D}}_t\mathbf{Z}_2\mathbf{T} & \psi_t\mathbf{I} \end{bmatrix} \boldsymbol{\alpha}_{t-1}^{\dagger} + \begin{bmatrix} \mathbf{c} \\ \exp(\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t) - \tilde{\mathbf{D}}_t\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t + \tilde{\mathbf{D}}_t\mathbf{Z}_2\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \tilde{\mathbf{D}}_t\mathbf{Z}_2 \end{bmatrix} \mathbf{R}\boldsymbol{\epsilon}_t \tag{9}$$

Given $\tilde{\boldsymbol{\alpha}}_t$, the LGAM approximating model is given by (8)-(9).

Consider the following iterative scheme:

(i) use $\tilde{\boldsymbol{\alpha}}_t, t = 1, \dots, T$, to construct the linearised state space model (8)-(9);

(ii) run the Kalman filter and smoother to obtain the smoothed estimates of the state,

$$\hat{\boldsymbol{\alpha}}_t^{\dagger} = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_t \\ \mathbf{Y}_{2t}^{\dagger} \end{bmatrix};$$

(iii) set $\tilde{\boldsymbol{\alpha}}_t = \hat{\boldsymbol{\alpha}}_t$;

(iv) iterate (i)-(iii) until convergence, i.e. until the Euclidean distance $\|\hat{\boldsymbol{\alpha}}_t - \tilde{\boldsymbol{\alpha}}_t\|$ is less than a specified tolerance value.

Hence, the Kalman filtering and smoothing equations run on the linearised model yield a new value $\hat{\boldsymbol{\alpha}}_t$, which replaces the previous trial value $\tilde{\boldsymbol{\alpha}}_t$ into the system matrices in (9), to give new approximating model. This process is iterated until convergence, in the sense specified above, and ensures that the final LG approximating model has the same conditional mode $\hat{\boldsymbol{\alpha}}_t$ as the original nonlinear one.

To illustrate this point, it should be recalled that for the linear Gaussian model the Kalman smoother provides the conditional mode (coincident with the mean) of the states $\boldsymbol{\alpha}_t^{\dagger}$, given the observations and the value $\tilde{\boldsymbol{\alpha}}_t$. Now, for a fixed $\tilde{\boldsymbol{\alpha}}_t$, taking the expectation of both sides of the transition equation for the elements in $\boldsymbol{\alpha}_t^{\dagger}$ conditional on the observations gives:

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_t &= \mathbf{T}\hat{\boldsymbol{\alpha}}_{t-1} + \mathbf{c} + \mathbf{R}\hat{\boldsymbol{\epsilon}}_t && (10) \\ \mathbf{Y}_{2t}^{\dagger} &= \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp(\mathbf{Z}_2\tilde{\boldsymbol{\alpha}}_t) + \tilde{\mathbf{D}}_t\mathbf{Z}_2(\hat{\boldsymbol{\alpha}}_t - \tilde{\boldsymbol{\alpha}}_t) && (11) \end{aligned}$$

where $\hat{\boldsymbol{\epsilon}}_t$ denotes the vector of smoothed disturbances. When the iteration converges, $\hat{\boldsymbol{\alpha}}_t \approx \tilde{\boldsymbol{\alpha}}_t$ and the equation (11) reduces to

$$\mathbf{Y}_{2t}^{\dagger} = \psi_t \mathbf{Y}_{2,t-1}^{\dagger} + \exp(\mathbf{Z}_2\hat{\boldsymbol{\alpha}}_t). \tag{12}$$

Now, (10) and (12) are exactly the equations that are satisfied by the conditional mode of the states in the true nonlinear model. The proof is straightforward: denoting $\boldsymbol{\alpha}$, $\mathbf{Y}_2^{\dagger}$, $\boldsymbol{\alpha}^{\dagger} = (\boldsymbol{\alpha}, \mathbf{Y}_2^{\dagger})$ and $\mathbf{y}$ the complete set of $\boldsymbol{\alpha}_t, \boldsymbol{\alpha}_t^{\dagger}$ and the observations for all times t, the conditional mode is the maximum of the conditional density $f(\boldsymbol{\alpha}^{\dagger}|\mathbf{y})$. However, since $\mathbf{y}$ is a linear transformation of the states $\boldsymbol{\alpha}^{\dagger}$, $f(\boldsymbol{\alpha}^{\dagger}|\mathbf{y}) = f(\boldsymbol{\alpha}^{\dagger}) = f(\boldsymbol{\alpha})f(\mathbf{Y}_2^{\dagger}|\boldsymbol{\alpha})$. The first density is linear and Gaussian whereas the second is unity, as $\mathbf{Y}_{2t}^{\dagger}$ is fully determined by its past value and $\boldsymbol{\alpha}_t$. Therefore $\hat{\boldsymbol{\alpha}}$ is such that $f(\hat{\boldsymbol{\alpha}})f(\mathbf{Y}_2^{\dagger}|\hat{\boldsymbol{\alpha}})$ is a maximum.

The iterative scheme is thus a particular case of the recursive conditional mode by extended Kalman filtering and smoothing proposed on Durbin and Koopman (1992, 2001) and Fahrmeir (1992). The solution is approximate, but the approximation can be made as accurate as needed.

The same argument can be exploited to show that the likelihood of the nonlinear model is equivalent to that of the approximating linear Gaussian model, once the iteration converge at $\hat{\boldsymbol{\alpha}}_t$. Hence, maximum likelihood estimation (MLE) and signal extraction are performed via linearizing the model, solving the conditional mode estimating equations and evaluating the likelihood of the optimized linear Gaussian model. As a matter of fact:

$$\begin{aligned} \ln f(\mathbf{y}) &= \int \ln f(\mathbf{y}, \boldsymbol{\alpha}^\dagger) d\boldsymbol{\alpha}^\dagger \\ &= \int \ln f(\mathbf{y}|\boldsymbol{\alpha}^\dagger) d\boldsymbol{\alpha}^\dagger + \int \ln f(\boldsymbol{\alpha}^\dagger) d\boldsymbol{\alpha}^\dagger \\ &= \int \ln f(\boldsymbol{\alpha}) d\boldsymbol{\alpha}^\dagger \end{aligned} \tag{13}$$

and the latter is approximated within the specified tolerance by the likelihood of the optimised linear Gaussian model.

An alternative representation that also uses the Gaussian likelihood of the linear approximating model considers the nonlinearity in the measurement equation, whereas the transition retains its linearity. Define the state vector $\boldsymbol{\alpha}_t^* = [\boldsymbol{\alpha}_t', \mathbf{q}_t', \mathbf{q}_{t-1}', \ldots, \mathbf{q}_{t-\delta+1}']'$ where $\mathbf{q}_t = \mathbf{Z}_2\boldsymbol{\alpha}_t$. The measurement equation for the aggregated time series is:

$$\mathbf{Y}_{2t} = \sum_{i=0}^{\delta-1} \rho_t \exp(\mathbf{q}_{t-j}),$$

for a suitable set of time-varying coefficients ρ_t.

This simplifies the inferences, at the expenses of a larger state vector, that features $N_2 \cdot (\delta - 1)$ elements in excess of the previous representation.

6 Illustrations

This sections presents an application implementing the methods described in the previous sections, concerning the estimation of an index of coincident indicator respectively for the U.S. economy. The U.S. index has a long tradition: the original SW model, that considered four monthly coincident indicators, has recently been extended by MM to include quarterly GDP figures, without taking into account the nonlinear temporal aggregation contraint, and it has been extended in various directions, see for instance Kim and Nelson (1999), who modelled the single index as a process with Markov switching in the mean.

The index is based on $N = 5$ coincident indicators that are the original four monthly indicators adopted by the Conference Board and considered by SW, plus quarterly real GDP, as in Mariano and Murasawa (2003). The series, listed below and displayed in Figure 1, are seasonally adjusted and transformed into logarithms.

- *IIP*: Index of Industrial Production, monthly, available for the sample period Jan. 1946 - Feb. 2003. Source: Board of Governors of the Federal Reserve System.

- *EMP*: Employment, number of employees on non-agricultural payrolls in thousands, monthly, available for the sample period Jan. 1946 - Feb. 2003. Source: Department of Labor, Bureau of Labor Statistics.

- *SLS*: Manufactured and trade sales in millions of chained 1996 dollars, monthly, available for the sample period Jan. 1959 - Jan. 2003. Source: Department of Commerce, Bureau of Census.

- *INC*: Personal income less transfer payments in billions of chained 1996 dollars, monthly, available for the sample period Jan. 1959 - Feb. 2003. Source: Department of Commerce, Bureau of Economic Analysis.

- *GDP*: Real Gross Domestic Product in billions of chained 1996 dollars, quarterly, available from the first quarter of 1947 to the first quarter of 2003. Source: Department of Commerce, Bureau of Economic Analysis.

The nonlinearity arises from the temporal aggregation of the GDP series, whose unobserved monthly growth rates depend on the single index, μ_t; thus $Y_{2t} =$ is scalar ($N_2 = 1$) and $\delta = 3$. Our application differs from previous ones not only because our model embodies the nonlinear temporal aggregation constraint, but also because it is formulated in the log-levels, rather than log-changes and we extend back the sample period to Jan.

1946, therefore entertaining 13 years of missing observations for *SLS* and *INC* and one year for *GDP*. As a by product of our modelling effort, not only disaggregated monthly GDP figures that satisfy the temporal aggregation constraint are made available, but also estimates of the missing values for the remaining series.

The estimation of model (8)-(9) was carried out maximising the likelihood obtained by the Kalman filter, with the modifications introduced by Koopman (1997) for dealing with initial diffuse effects, that result from the nonstationarity of some of the state elements. All the computations were carried out using Ox 3.20 by Doornik (2001) and the package SsfPack 3.0 beta (see Koopman, Shephard and Doornik, 1999, 2002). For the common factor, we adopt the SW identification assumption, that sets the variance of the disturbances σ_η^2 equal to 1 in (1); moreover, the common factor and the idiosyncratic components have an ARIMA(2,1,0) representation, that is we set $p = p_i = 2, i = 1, \ldots, 5$.

The parameter estimates, along with their asymptotic standard errors, are presented in Table 1. The log-likelihood for the approximating model is $\mathcal{L} = 10015.53$. The estimated factor loadings are all positive and significantly different from zero, as expected. The estimates of the autoregressive coefficients d_{i1} and d_{i2} for the monthly indicators, that regulate the dynamics of the idiosyncratic component are fairly similar to those obtained by SW and MM. The differences are explained not only by the fact that we entertain a nonlinear model, but are also due to the larger sample period considered, starting in 1946 in our application and the revisions occurred in the indicators over time. The autoregressive coefficients of μ_t show an higher value at lag two (0.1856) with respect to SW and MM estimates (0.032 and 0.08 respectively). With respect to *GDP* the first autoregressive term is positive, whereas it is slightly negative for MM (-0.04).

Table 2 presents some model diagnostics based on the Kalman filter innovations. In particular, the statistics that we consider are the Box-Ljiung statistics $Q(15)$ and $Q(25)$ based, respectively, on 15 and 25 autocorrelations, the Bowman-Shenton normality test ($Norm$) and the heteroscedasticity statistic $H(h)$, where $h = 229$ for the monthly indicators *IIP*, *EMP*, *SLS*, and *INC* and $h = 76$ for *GDP*. The results suggests a satisfactory specifications for all the equations. The high values for *IIP* and *EMP* in the Normality test arise in connection with a limited number of outliers occurring in the first two decades. However, we did not make any adjustments for those, nor we changed the model specification. Overall, the model shows a general good fit and our interest goes much more in the reliability of the method in determining the business cycle and in distributing the aggregated values.

The estimates $\tilde{\mu}_t$ of the coincident index, conditional on the full multivariate sample, have been obtained using a fixed interval smoother and are presented in figure 2. At the modelling stage, we constrained the drift of μ_t to be equal to zero, since we could not identify six independent drifts terms from five series, without introducing a linear constraints among them. The identification of the drift for the common single index can be done ex-post (as in SW), and it is a crucial issue for the interpretability of the component. Also, we constrained the variance of the disturbances η_t to be equal to one, since we could not identify the scale of the common factor without restricting one factor loading. Now, location and scale are crucial for the interpretability of the index, especially in terms of business cycle features: for instance, recession probabilities in the classical sense crucially depend on this two parameters, a point that is clearly stated in Pagan (2002). For this purpose we set the drift equal to that of monthly *GDP*, that is obtained from the model estimates as follows: $\tilde{b} = \left(1 - \tilde{d}_{GDP,1} - \tilde{d}_{GDP,2}\right)^{-1} \tilde{\beta}_{GDP}$. Moreover the index is rescaled by multiplying it for the GDP loading on the common factor. In conclusion, our index of coincident indicators, that we denote CI_t, is calculated as follows:

$$CI_t = \tilde{\theta}_{GDP}\tilde{\mu}_t + \tilde{b}t.$$

The plot of the $\mathsf{E}[\exp(CI_t)] = \exp(CI_t + 0.5V_t)$, where V_t is the conditional estimation error variance of CI_t computed by the smoothing algorithm, is presented in Figure 3, along with the monthly estimates of GDP in their original levels, consistent with the quarterly observed totals, and the SW's experimental index XCI. We notice in passing that the latter is the cumulation of the filtered estimates of $\Delta\mu_t$ (SW carefully discuss the identification of the drift of this component). It clearly visible that XCI emphasizes much more the amplitude of cycles; this is so on the one hand since the latter does not include *GDP* in its construction, and, more importantly, its scale has not been reduced to match that of the common component of *GDP*.

7 Conclusion

The paper has developed a novel solution to the problem of modelling time series subject to a nonlinear temporal aggregation constraints. This situation arises within a dynamic factor model for a multivariate time series whose components are observed at different frequencies (quarterly and monthly), and are modelled in their logarithms.

An illustrations was presented referring to the U.S. economy in which the traditional set of monthly coincident indicators is augmented by Gross Domestic Product, which represents the main coincident indicator, but it is available only at the quarterly frequency. From the empirical standpoint, the main contribution of the paper is to provide monthly GDP estimates that are consistent with the quarterly totals.

The solution is simple to implement since it involves determining the linear Gaussian approximation that has the same conditional mode, which is performed in practice by iterating the Kalman filtering and smoothing equations. Although it is based on an approximation, the latter can be made as accurate as it is necessary, so that we can regard our methods as providing an exact treatment of disaggregation under a nonlinear constraint.

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Table 1: Index of coincident indicators for the U.S.: parameter estimates and asymptotic standard errors

Parameters	*IIP*	*EMP*	*SLS*	*INC*	*GDP*
$\theta \times 100$	*0.853*	*0.245*	*0.677*	*0.320*	*0.372*
	(0.035)	*(0.011)*	*(0.041)*	*(0.021)*	*(0.021)*
$\beta \times 100$	*0.425*	*0.093*	*0.489*	*0.267*	*0.424*
	(0.106)	*(0.017)*	*(0.100)*	*(0.035)*	*(0.154)*
d_{i1}	*-0.174*	*0.166*	*-0.446*	*-0.014*	*0.175*
	(0.062)	*(0.047)*	*(0.049)*	*(0.051)*	*(0.366)*
d_{i2}	*-0.283*	*0.295*	*-0.221*	*0.056*	*-0.660*
	(0.061)	*(0.059)*	*(0.047)*	*(0.052)*	*(0.172)*
$\sigma_{\eta^*} \times 100$	*0.570*	*0.181*	*0.776*	*0.296*	*0.396*
	(0.029)	*(0.001)*	*(0.003)*	*(0.011)*	*(0.130)*

$$\underset{}{(1 - \underset{(0.049)}{0.3382}L - \underset{(0.051)}{0.1856}L^2)}\,\Delta\mu_t = \eta_t, \quad \eta_t \sim N(0,1)$$

Note: standard errors in parenthesis.

Table 2: Diagnostics for the US model

Tests	*IIP*	*EMP*	*SLS*	*INC*	*GDP*
$Q(15)$	*22.131*	*31.953*	*22.767*	*27.916*	*10.520*
$Q(25)$	*42.736*	*60.365*	*42.010*	*33.964*	*29.153*
$Norm$	*163.491*	*2485.725*	*15.444*	*31.956*	*28.235*
$H(h)$	*0.321*	*0.204*	*3.794*	*2.931*	*0.319*

Note: *Q(15)* and *Q(25)* are the Box-Ljiung statistics based, respectively on *15* and *25* residual autocorrelations, *Norm* is the Bowman-Shenton Normality test and *H(h)* is the test for heteroscedasticity (*h=76* for *GDP* and *h=229* for the other series) .

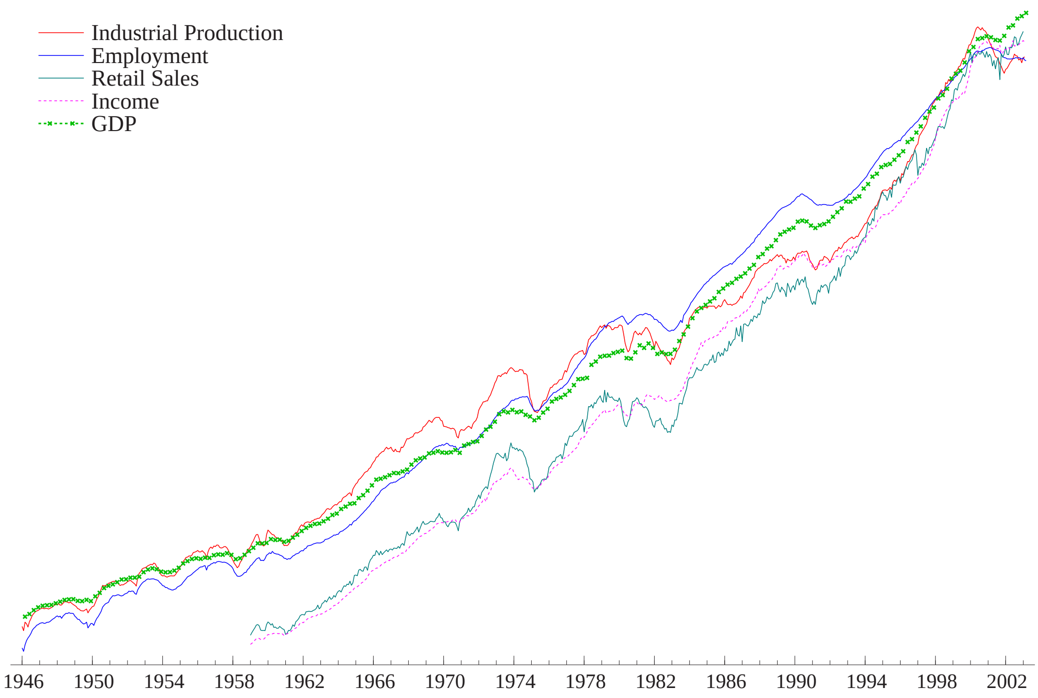

Figure 1: Coincident indicators for the U.S.

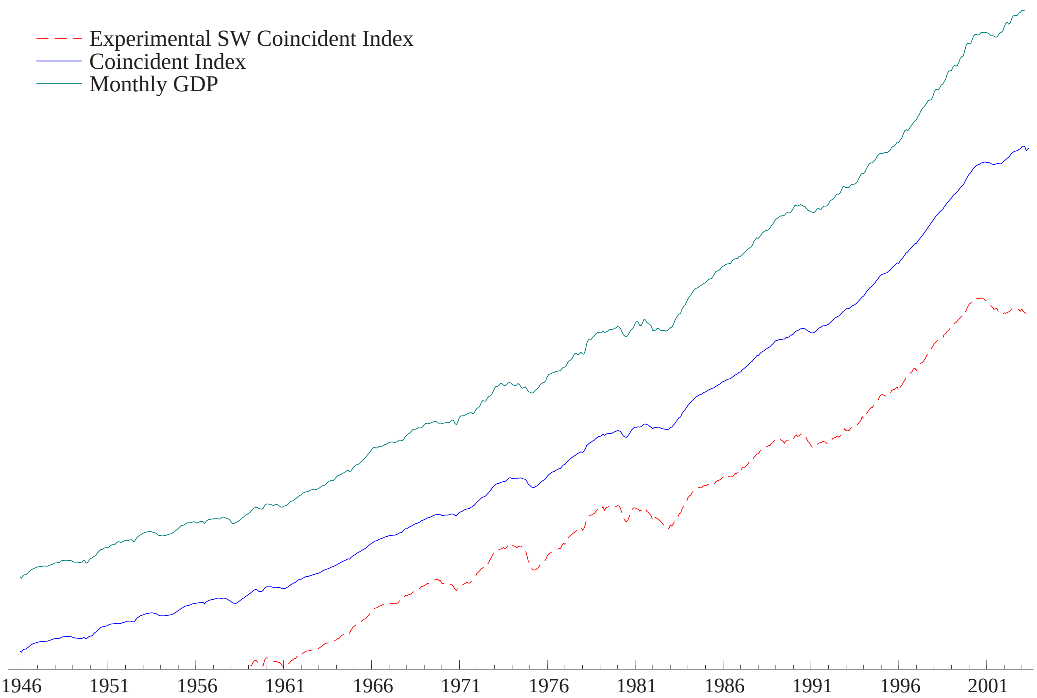

Figure 2: Index of coincident indicators and monthly GDP for the U.S.