# A Self-Consistent Model for the Forward Price Dynamics

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# ABSTRACT

We consider mean-reverting stochastic processes and build a self-consistent model for forward price dynamics and their applications in power industries. This model with stochastic volatility of the forward price is built using the ideas and equations of stochastic differential geometry in order to close the system of equations for the forward price and its volatility. Stationary distributions for the forward price volatility are found analytically as well as the forward price curves in the one factor case. We consider two models for regular forward price volatility.

1) Pure exponential two parameter model with zero asymptotic at $T \to \infty$:
   $\Sigma(t,T) = \sigma_1 e^{-\alpha(T-t)}$

2) Three parameter model with non-zero asymptotic at $T \to \infty$
   $\Sigma(t,T) = \sigma_1 e^{-\alpha(T-t)} + \sigma_0$

The first model is a toy one although it can be used in the case of long terms, the second one is quite reliable for short terms. Those models will also play a role of initial conditions for a stochastic process described forward price volatility.

We compare our results with those known from the literature.

## I. Forward Prices and Their Modeling

***Definition1.*** A *forward contract* is a particularly simple derivative and is an agreement to buy (a *long position*) or to sell (a *short one*) an asset at a certain future time (*maturity date* $T$) for a certain price (the *delivery price* $K$).
At time the contract is initiated the delivery price should be such that the contract value for both parties is zero. The contract is obligatory.

***Definition 2.*** The *forward price* $F(t,T)$ for a certain contract is defined as the delivery price which would make the contract have zero value.
The forward price and the delivery price are equal at the time the contract is entered into. As time passes, they go apart since pre-specified initially delivery price is constant. Therefore we can think of the forward price as the delivery price at current time $t$.

There is a very well-known formula relating the forward price and the spot price Hull (1993)

$$F(t,T) = S(t)e^{(r+u-y)(T-t)} \qquad \text{(I-1)}$$

Where $r$ is a risk-less interest rate, $u$ is a storage rate and $y$ a convenience yield. Both the storage rate and especially the convenience yield are usually unknown functions and the convenience yield can be a stochastic process.

Following Cortazar & Schwartz (1994) we will describe the dynamics of the forward price by the equation

$$\frac{dF(t,T)}{F} = \sum_{p=1}^{n} \Sigma_p(t,T)dW^p(t) \qquad \text{(I-2)}$$

where $\Sigma_p(t,T)$ is a volatility corresponding to a $p$-th random factor described by the Wiener generator $dW^p(t)$. So model (I-2) describes the n-factor dynamics of the forward curve $F(t,T)$.

We assume that interest rates are deterministic and future prices are equal to forward prices (see, e.g. Hull (1993)). In (2) we have *n* independent sources of uncertainty that drive the evolution of the forward curve *F*(*t*,*T*).

By integrating (2) we have (using Ito's lemma)

$$F(t,T) = F(0,T)\exp\left\{\sum_{i=1}^{n}\left[-\frac{1}{2}\int_0^t \Sigma_i(\tau,T)^2\,d\tau + \int_0^t \Sigma_i(\tau,T)\,dW^i(\tau)\right]\right\} \qquad \text{(I-3)}$$

Then for the spot price *S*(*t*) we have by setting $T = t$

$$S(t) = F(0,t)\exp\left\{\sum_{i=1}^{n}\left[-\frac{1}{2}\int_0^t \Sigma_i(\tau,t)^2\,d\tau + \int_0^t \Sigma_i(\tau,t)\,dW^i(\tau)\right]\right\} \qquad \text{(I-4)}$$

It means that the natural logarithm of the spot price (as well as the forward price) is normally distributed at time *T* given the forward price initially at time zero such that

$$\ln(S(T)) \approx N\left[\{\ln(F(0,T)) - \frac{1}{2}\sum_{i=1}^{n}\int_0^T \Sigma_i(\tau,T)^2\,d\tau\},\ \sum_{i=1}^{n}\int_0^T \Sigma_i(\tau,T)^2\,d\tau\}\right]$$

By differentiating (4) over $t$ we have got the stochastic differential equation for the spot price

$$\frac{dS(t)}{S(t)} = \left\{\frac{\partial \ln F(0,t)}{\partial t} - \sum_{i=1}^{n}\left[\int_0^t \Sigma_i(\tau,t)\frac{\partial \Sigma_i(\tau,t)}{\partial t}\,d\tau - \int_0^t \frac{\partial \Sigma_i(\tau,t)}{\partial t}\,dW^i(\tau)\right]\right\}dt$$
$$+\sum_{i=1}^{n}\Sigma_i(t,t)\,dW^i(t) \qquad \text{(I-5)}$$

The term in the curled parentheses can be interpreted as an equivalent to the sum of the deterministic risk-less rate of interest $r(t)$ and a convenience yield $y(t)$ *which in general should be stochastic.*

Many well-known models are special cases of this general approach. E.g. Schwartz's (1997) model for the commodity price dynamics used a *single factor mean-reverting model:*

$$dS = \alpha[\mu - \ln(S)]S\,dt + \sigma S\,dW \qquad \text{(I-6)}$$

Let us consider single factor model (2) for *F*(*t*,*T*), too

$$\frac{dF(t,T)}{F(t,T)} = \Sigma(t,T)dW(t) \quad \text{and} \quad F(t,t) = S(t) \qquad \text{(I-7)}$$

Now a question arises what kind of model we will adopt for the volatility $\Sigma(t,T)$.

Let us assume that the volatility is a regular function of time of the form

$$\Sigma(t,T) = \sigma_1\, e^{-\alpha(T-t)} \qquad \text{(I-8)}$$

This function is very popular and convenient to handle integrals with. Then after a simple though tedious calculations we can find the forward price curve solving equations (4)-(6) together.

$$F_{1R}(0,T) = \exp\left\{ e^{-\alpha T}\ln S(0) + \mu(1-e^{-\alpha T}) - \frac{\sigma_1^2}{4\alpha}(1-e^{-\alpha T})^2 \right\} \qquad \text{(I-9)}$$

The equation (5) can be solved if the following condition (similar to the “risk-less condition” in the Black-Scholes theory) holds:

$$\partial_T \Sigma(t,T) = -\gamma\Sigma(t,T)$$

or in more general form

$$\partial_T \Sigma(t,T) = -\gamma\Sigma(t,T) + f(t,T) \qquad \text{(I-10)}$$

where $f(t,T)$ is a known function.

Although three-parameter function (8) is very simple and popular but in reality we rather have a four-parameter function of the form, see Clewlow and Strickland (1999), Figure 2:

$$\Sigma(t,T) = ae^{-\alpha(T-t)} + b \tag{I–8a}$$

After the similar calculations we obtain the following formulae for the average forward price and its volatility:

$$< \ln F_{2R}(0,T) >= e^{-\alpha T} S(0) + \mu(1 - e^{-\alpha T}) - \frac{1}{4\alpha}\left[a^2(1-e^{-\alpha T})^2 - 2b^2(\alpha T - 1 + e^{-\alpha T})\right] \tag{I-9a }$$

$$Var\left[\ln F_{2R}(0,T)\right] = \frac{b^2}{2\alpha}\left[2e^{-\alpha T}(1-\alpha T + \alpha^2 T^2) - e^{-2\alpha T}(2+\alpha^2 T^2)\right] \tag{I-9b}$$

Now it’s well known that the volatilities in (2) are stochastic processes themselves. What kind of stochastic processes they could be? To answer this question we resort to the stochastic differential geometry Makhankov (1995, 1997)
As a result we obtain a self-consistent model described by the system of stochastic equations.

## Stochastic differential geometry

The equations of stochastic differential geometry that describes the Brownian motion (diffusion) in a curved space (manifold) read

$$dX^i(t) = \sum_{q=1}^{n} \Sigma^i_q dW^q(t)$$

$$d\Sigma^i_q = -\sum_{j,k=1}^{m} \Gamma^i_{jk}\Sigma^j_q dX^k \tag{I-11}$$

Where $X^i$ is an m-dim vector (a point in an m-dim curved space), $\Sigma^i_q$ is a matrix of rotating operator and can be constructed of $m$ vectors which set up a natural frame on a patch of the bundle and $\Gamma^i_{jk}$ is a conexion coefficients through which a curvature of the space is given. The first equation describes an elementary shock the Brownian particle undergoes due to collision with the stochastic background.

While moving from one patch to another along with the particle, this frame changes the orientation.

The total change of a vector, ***B*** due to moving from one point to another consists of two pieces

$$\delta\vec{B} = d\vec{B} + \Gamma\vec{B}d\vec{X} \qquad (\text{I-12})$$

Where the first term is the differential along the path

$$d\vec{B}(t) = \frac{\partial\vec{B}}{\partial X^i} dX^i(t) \qquad .$$

and the second allows for a change of the frame orientation. Since the matrix $\Sigma$ consists of $n$ vectors it is transformed following the same rule

$$\delta\Sigma = d\Sigma + \Gamma\Sigma d\vec{X}(t) \qquad (\text{I-13})$$

Now the fair game rule (no arbitrage opportunity, same action same response) means that the total change of $\Sigma$ should vanish

$$\delta\Sigma = 0$$

or

$$d\Sigma = -\Gamma\Sigma\, d\vec{X}(t)$$

which along with the equation for the elementary shock

$$d\vec{X}(t) = \Sigma\, d\vec{W}$$

gives the equations of Stochastic Differential Geometry. Let us stress that the equations are written in *Stratonovich* differentials.

So we have got the model that describes pure Brownian motion in a curved space. From the other hand from general relativity we know that a curvature may be considered as force. So what kind of force it is in our case. The dynamics of the forward price logarithm is given by the equation

$$d\ln(F(t,T)) = -\frac{1}{2}\sum_{q=1}^{n}\Sigma^2{}_q(t,T)dt + \sum_{q=1}^{n}\Sigma_q(t,T)\, dW^q(t) \qquad (\text{I-14})$$

in *Ito* differentials. If we assume the internal term space of the model being discrete (what is true in reality) we have got

$$F(t,T) = F(t,kT) = F^k(t)$$

Now denoting

$$X^k(t) = \ln F^k(t) \qquad \text{(I-15)}$$

we come to the equation

$$dX^i(t) = -\frac{1}{2}\sum_{q=1}^{n} {\Sigma_q^i}^2(t)dt + \sum_{q=1}^{n} \Sigma_q^i dW^q(t), \quad i \in (1,\ldots,m) \qquad \text{(I-16)}$$

written in *Ito* differentials.

If we wish that the dynamical model of forward price should correspond to the pure Brownian motion in the curved manifold we have to equate eq. (I-16) to the first equation of system (I-11) also written in the *Ito* differentials,

$$dX^i(t) = \sum_{q=1}^{n} (\Sigma_q^i + \frac{1}{2} d\Sigma_q^i) dW^q(t) = \sum_{q=1}^{n} \Sigma_q^i dW^q(t) - \frac{1}{2} \sum_{j,k,q} \Gamma_{jk}^i \Sigma_q^k \Sigma_q^j dt$$

(I-17)

Then we have the equation for self-consistency of the model

$$\sum_{q=1}^{n} {\Sigma_q^i}^2(t)dt = \sum_{q}^{n} \sum_{j,k}^{m} \Gamma_{jk}^i \Sigma_q^k \Sigma_q^j dt \qquad \text{(I-18)}$$

Resolving this equation with respect to

$$A_{k,q}^i = \Gamma_{jk}^i \Sigma_q^j$$

we obtain

$$A_{k,q}^i = \Sigma_q^i \, \delta_k^i$$

Substituting this equation into the second one of (I-11) we come to the equation

$$d\Sigma_p^i(t) = -\Sigma_p^i \sum_{q=1}^{n} \Sigma_q^i dW^q(t) \qquad \text{(I-19)}$$

***Now our system is closed and self-consistent since the curvature of the term space is defined by the "force term" (the trend) in the equation for the price dynamics.***

So we have to solve the following system of equations

$$dX^i(t) = \sum_{q=1}^{n} \Sigma_q^i dW^q(t)$$

$$d\Sigma_p^i(t) = -\Sigma_p^i \sum_{k=1}^{n} \Sigma_q^i dW^q(t)$$

that in *Ito* differentials read

$$dX^i(t) = -\frac{1}{2}\sum_{q=1}^{n} \Sigma_q^{i\,2}(t)dt + \sum_{q=1}^{n} \Sigma_q^i dW^q(t)$$

$$d\Sigma_q^i = \Sigma_q^i \sum_{p}^{n} (\Sigma_p^i)^2 dt - \Sigma_q^i \sum_{p}^{n} \Sigma_p^i dW^p \qquad \text{(I-20)}$$

From the first equation of (I-20) we infer that $X$ has a Gaussian distribution. What about the volatility?

Let us consider a one-factor reduction of the model. It makes sense since as is well-known some, may be even many power markets as well as financial ones show almost one-factor behavior (principal component analysis gives from 80% to even 90% of the total contribution to the first component). Clewlow & Strickland (1999), Wilmott (2001).

So, one factor: $dW^p = dW$ and $\Sigma^i = Y$ then

$$dY = Y^3 dt - Y^2 dW \qquad \text{(I-21)}$$

and the Focker-Planck equation for the transition probability $\rho$ reads

$$\partial_t \rho = \partial_y(-y^3 + \frac{1}{2}\partial_y y^4)\rho \qquad \text{(I-22)}$$

Let us consider stationary solutions of (I-22). Then we have

$$\partial_y(-y^3 + \frac{1}{2}\partial_y y^4)\rho = 0$$

with a solution

$$\rho = \frac{cy + a}{y^3} \qquad \text{(I-23)}$$

If we consider the related *Stratonovich* process

$$d\Sigma_q^i = -\Sigma_q^i \sum_p^n \Sigma_p^i dW^p$$

or for a single-factor, single-term process $Z = \Sigma^i$ we have

$$dZ = -Z^2 dW \qquad \text{(I-24)}$$

with the FP equation

$$\partial_t \rho = \frac{1}{2}\partial_z (z^2 \partial_z z^2)\rho$$

and stationary solutions

$$\rho = \frac{a}{z^3} \qquad \text{(I-25)}$$

Then we see that both processes have similar distributions if $y$ is sufficiently small

$$y << \frac{a}{c}$$

It was the *Stratonovich* process. For the *Ito* process we have eq. (I-21). The mean can be estimated from the trend term by the following reasoning: taking average of the equation we have for $\Sigma =< \Sigma > +s$ and $< s >= 0$

$$< d\Sigma >=< \Sigma^3 >=< s+ < \Sigma >>^3 = 3 < s^2 >< \Sigma > + < \Sigma >^3$$

Where the variance $< s^2 >\approx t < \Sigma >^4$ and the first term in the equation can be neglected. Then since $< d\Sigma >=< \Sigma_{t+1} - \Sigma_t >= d < \Sigma >$ we come to

$$d < \Sigma >=< \Sigma >^3 \ dt$$

with the solution

$$< \Sigma >= \sigma\sqrt{\frac{1}{1-t\sigma^2}} \approx \sigma(1+\frac{1}{2}t \ \sigma^2)$$

where we have

$$\sigma =< \Sigma(0,T) >= \sigma_1 e^{-\alpha T} \qquad \textit{model} \ \text{(I.8)}$$

$$\sigma =< \Sigma(0,T) >= a e^{-\alpha T} + b \qquad \textit{model} \ \text{(I.8a)}$$

Now armed with the above knowledge we can calculate the forward price curve. In order to obtain analytical estimate we restrict ourselves to “short time” horizons:

$$\sigma^2 t << 1 \qquad \text{(I-26)}$$

and consider only first two initial terms in the asymptotic expansion.

In the previous part we consider the statistical properties of the model and short time horizons. In what follows we study the dynamics of the model in more detail.

## II. First equation of the SDG model

Let us consider again the single-factor model (I-7)

$$\frac{dF}{F} = \Sigma(t,T)dW(t)$$

Or using Ito’s lemma

$$d\ln F(t,T) = -\frac{1}{2}\Sigma^2(t,T)dt + \Sigma(t,T)dW(t)$$

Integrating once we have

$$\ln\frac{F(t,T)}{F(0,T)} = -\frac{1}{2}\int_0^t \Sigma^2(u,T)du + \int_0^t \Sigma(u,T)dW(u)$$

or

$$F(t,T) = F(0,T)\exp\{-\frac{1}{2}\int_0^t \Sigma^2(u,T)du + \int_0^t \Sigma(u,T)dW(u)\} \qquad \text{(II-1)}$$

This solution is defined so far accurate to an arbitrary function *F*(0,*T*). To restrict this freedom we can specify a random process for the spot price *S*(*t*). Now since *S*(*t*)= *F*(*t*,*t*), knowing the equation for *S*(*t*) gives us the equation for *F*(0,*T*) through the parameters involved in the eqn. for *S*(*t*).

Let us consider a mean-reverting process for *S*(*t*), viz.

$$\frac{dS}{S} = \alpha(\mu - \ln S)dt + \sigma(t)dW(t) \qquad \text{(II-2)}$$

From the other hand eqn. (II-1) gives

$$S(t) = F(0,t)\exp\{-\frac{1}{2}\int_0^t \Sigma^2(u,t)du + \int_0^t \Sigma(u,t)dW(u)\} \qquad \text{(II-3)}$$

i.e. ln*S* is normally distributed with

mean = $\ln F - \frac{1}{2}\int_0^t \Sigma^2(u,T)du$

dispersion = $\Sigma(t,t)$

Also from eqn. (II-3) taking the log

$$\ln S(t) = \ln F(0,t) - \frac{1}{2}\int_0^t \Sigma^2(u,t)du + \int_0^t \Sigma(u,t)dW(u) \qquad \text{(II-4)}$$

Then by differentiating over *t* one has

$$d\ln S(t) = [\frac{\partial \ln F(0,t)}{\partial t} - \frac{1}{2}\Sigma^2(t,t) - \int_0^t \Sigma(u,t)\Sigma_t(u,t)du$$
$$+\int_0^t \Sigma_t(u,t)dW(u)]dt + \Sigma(t,t)dW(t)$$

Easy to check out that from Ito's lemma follows that

$$d\ln S(t) + \frac{1}{2}\Sigma^2(t,t)dt = \frac{dS}{S}$$

Therefore

$$\frac{dS(t)}{S(t)} = [\frac{\partial \ln F(0,t)}{\partial t} - \int_0^t \Sigma(u,t)\Sigma_t(u,t)du + \int_0^t \Sigma_t(u,t)dW(u)]dt$$
$$+\Sigma(t,t)dW(t) \qquad \text{(II-5)}$$

Now if the spot process underlying the forward price dynamics is defined by eqn. (II-2) we have the self-consistent system of equations:

$$\sigma(t) = \Sigma(t,t)$$

(II-6)

$$\alpha(\mu - \ln S) = \frac{\partial \ln F(0,t)}{\partial t} - \int_0^t \Sigma(u,t)\Sigma_t(u,t)du + \int_0^t \Sigma_t(u,t)dW(u)$$

(II-7)

Rewrite eqn. (II-4) in the form

$$\int_0^t \Sigma(u,t)dW(u) = \{\ln S(t) - \ln F(0,t)\} + \frac{1}{2}\int_0^t \Sigma^2(u,t)du$$

(II-8)

We can easily solve the system of equations (II-5), (II-7) and (II-8) if

$$\Sigma_t(u,t) = -\alpha\Sigma(u,t)$$

(II-9)

or more general

$$\Sigma_t(u,t) = -\alpha\Sigma(u,t) + f(t)$$

(II-10)

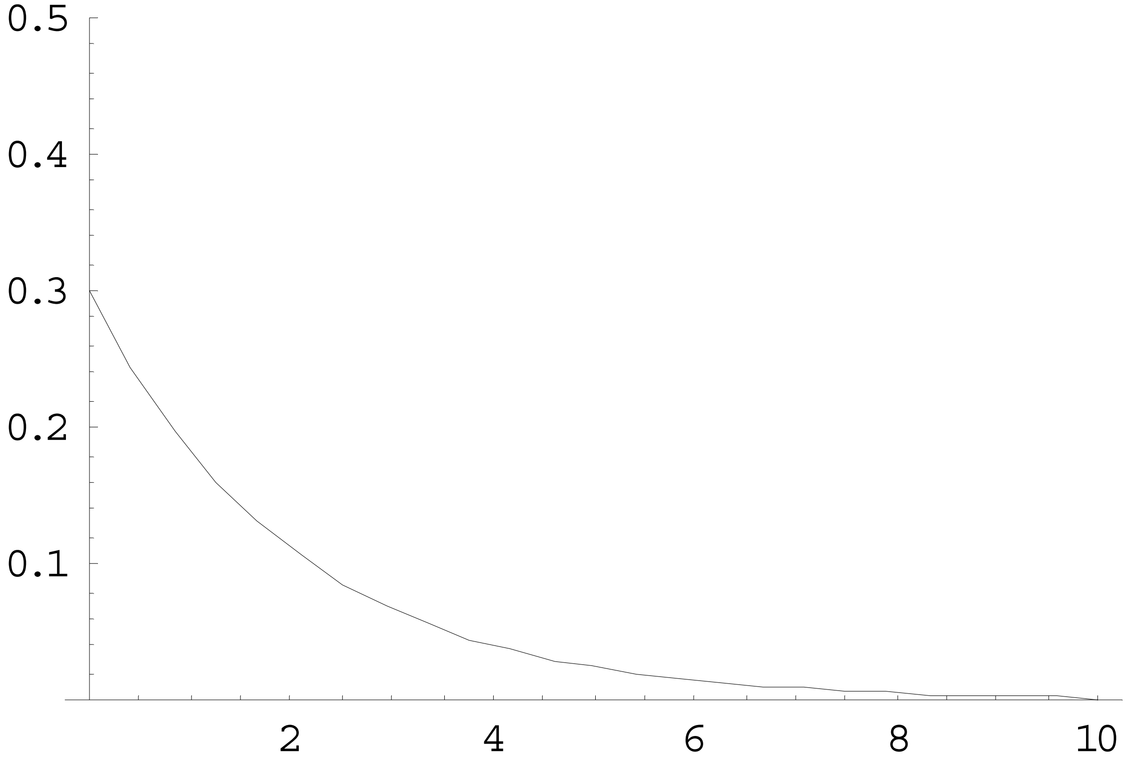
where *f*(*t*) is a known function of *t*. Those conditions are additional necessary for solvability of the whole problem. They look very plausible for they mean that the volatility of the forward price decays from one level to another or zero. If we substitute eqn (II-9) into (II-7) and use (II-8) we come to

$$\frac{\partial \ln F(0,t)}{\partial t} + \alpha \ln F(0,t) = \alpha\left(\mu - \frac{1}{2}\int_0^t \Sigma^2(u,t)du\right) \equiv \Phi_1(t)$$

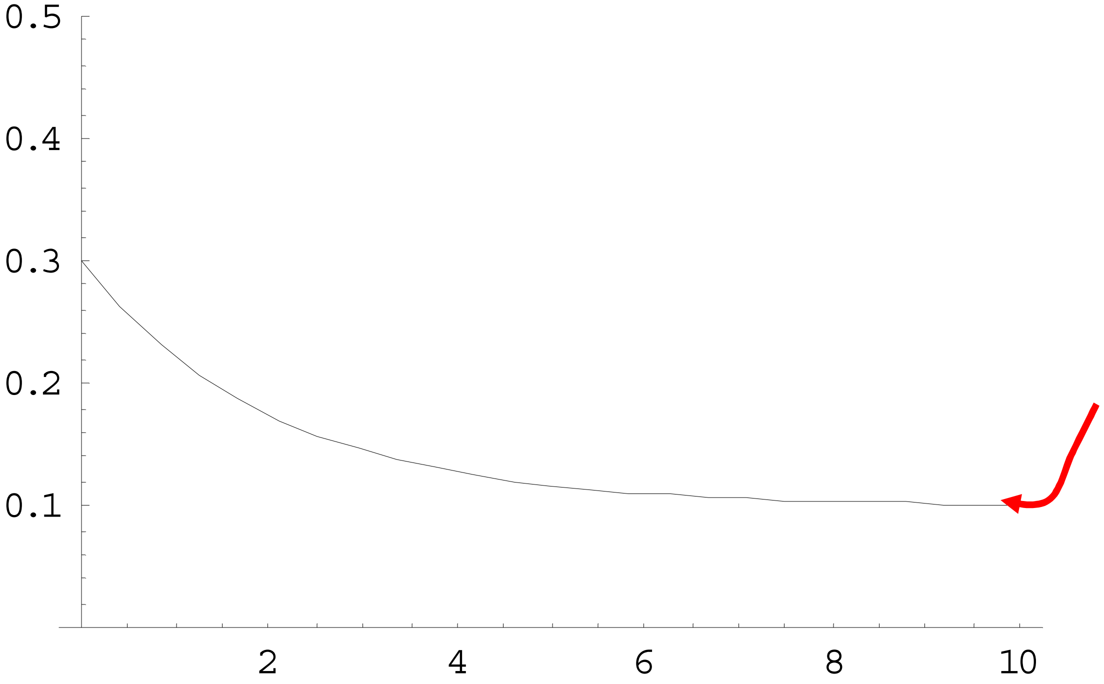
(II-11)

Below we give the graphs of modeling volatility curves and a real one.

$\Sigma(0,T) = 0.3Exp(-0.5T)$



$\Sigma(0,T) = 0.2Exp(-0.5T) + 0.1$



What we have in reality (the second curve from above), see Clewlow & Strickland (1999).

In general case (II-10) we have

$$\Phi(t) = \alpha\left(\mu - \frac{1}{2}\int_0^t \Sigma^2(u,t)du\right) + \int_0^t \Sigma(u,t)f(u)du - \int_0^t f(u)dW(u)$$

(II-12)

Equation (II-11) along with (II-12) can be readily solved provided we know the forward price volatility $\Sigma(t,T)$.

The solution (obtained by the variation of constant method) is

$$\ln F(0,t) = e^{-\alpha t}\{\int_0^t e^{\alpha u}\Phi(u)du + const\} \qquad \text{(II-13)}$$

Since at $t = 0$ ln$F$(0,0) = ln$S$(0) and

$$\ln F(0,t) = e^{-\alpha t}\{\int_0^t e^{\alpha u}\Phi(u)du + \ln S(0)\}$$

We see $\mu$ not necessarily be a constant. It can be a function of time.

For example: if the forward price volatility

1) is a regular function of time

$$\Sigma(t,T) = \sigma_1 e^{-\alpha(T-t)}$$

2) $\mu(t) = \mu_1 e^{\kappa t}$ (a constant $\kappa$ can be both negative or positive)

Then the integral is exactly evaluated as

$$F(0,T) = \exp\{e^{-\alpha T}\ln S(0) + \frac{\alpha\mu_1}{\alpha+\kappa}(e^{\kappa T} - e^{-\alpha T}) - \frac{\sigma_1^2}{4\alpha}(1 - e^{-\alpha T})^2\}$$

`

(II-15)

## Second Equation of the SDG model

This equation is self-consistent and can be separately analyzed. Under the same assumption (single-factor model) for Stratonovich process it reads, see eqn.(I-24)

$$d\Sigma_{str}(t,T) = -\Sigma_{str}^{2}(t,T)\,dW(t) \tag{1.24}$$

and

$$d\Sigma_{I} = \Sigma_{I}{}^{3}dt - \Sigma_{I}{}^{2}dW \tag{I-21}$$

for the Ito process.

From eqns. (I-23) and (I-25) one can see that the solutions to both FP equations are identical if $c<<a$ or e.g. when $c = a$. It means that in our case the *Stratonovich* process distributions are a subclass of the more general Ito's distributions.

Since for *Stratonovich* processes we have the conventional calculus we get

$$\Sigma_{str}(\tau,t) = \frac{\sigma(0,t)}{1+\sigma(0,t)W(\tau)}$$

or

$$\Sigma_{str}(t,T) = \frac{\sigma(0,T)}{1+\sigma(0,T)W(t)} \tag{II-16}$$

And

$$\partial_{t}\Sigma_{str}(\tau,t) = \frac{\sigma_{t}(0,t)}{(1+\sigma(0,t)W(\tau))^{2}}$$

In the first equations (1-25) and (1-26) $W(t)$ is a standard Wiener process with mean-less and unity-variance Gaussian distribution. So we can express $W(t)$ as a function of $\Sigma^i$

$$W(t) = \frac{1}{\Sigma^i(t)} - \frac{1}{\Sigma^i(0)} \qquad \text{(II-17)}$$

Also it is easy to calculate the mean and variance of for small $t$ (short horizons)

$$E[\Sigma^i(t,T)] \approx \sigma^i(0,T)\{1 + t\,\sigma^i(0,T)^2\}$$

$$Var[\Sigma^i(t,T)] \approx t\,\sigma^i(0,T)^4\{1 + 6t\,\sigma^i(0,T)^2\}$$

It should be mentioned that the series over $t$ are asymptotic and in principle are divergent and, strictly speaking, the only point of convergence is $t = 0$ even without its neighborhood. Also as could be expected the results are independent of a sign of the random term.

Now for the Stratonovich process we can go even further. Due to (II-18) we calculate the distribution for $\Sigma^i(t,T)$ by means of the formula

$$\rho(w)dW = \rho(1/\Sigma)\frac{dw}{d\Sigma}d\Sigma \equiv \rho(\Sigma)d\Sigma$$

And for $\rho(w)$ is a Gaussian distribution we have

$$\rho(\Sigma) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(1/\Sigma - 1/\Sigma_0)^2}{2t}\right)\frac{d\Sigma}{\Sigma^2}$$

a distribution for a reciprocal of W(t).

## *Model with two exponential parameters*

Assume that

$$\Sigma_t(u,t) = -\alpha\Sigma(u,t)$$

i.e. $\sigma(0,t) = \sigma_1 e^{-\alpha t}$

Then

$$\partial_t \Sigma_{str}(\tau,t) = -\alpha \frac{\sigma(0,t)}{(1+\sigma(0,t)W(\tau))^2} = -\frac{\alpha\Sigma(\tau,t)}{(1+\sigma(0,t)W(\tau))} \qquad \text{(II-18)}$$

Assume also that

$$\sigma(0,T)W(T) << 1 \qquad \text{(II-19)}$$

The last condition is very essential for evaluations. This is because

$$\sigma(0,T)W(t) = \sigma_1 e^{-\alpha T} W(t) \Rightarrow^{t \le T} \sigma_1 e^{-\alpha t} W(t) e^{-\alpha(T-t)} \le \sigma_1 e^{-\alpha t} W(t) \square \sigma_1 e^{-\alpha t}\sqrt{t}$$

There are two cases:

1) $\alpha T << 1$ then $\sigma_1^2 T << 1$

2) $\alpha T >> 1$, $\sigma_1^2 T$ is arbitrary

Now we have

$$\begin{aligned} &\partial_t \Sigma(u,t) = -\alpha\Sigma(u,t) + f(u) \\ &f(u) = \alpha W(u)\sigma^2(0,t)[1-2\sigma(0,t)W(u)] \end{aligned} \qquad \text{(II-20)}$$

And

$$\frac{1}{\alpha}\Phi(u) = \mu + 2\sigma^3(0,u)\int_0^u W(\tau)d\tau - \frac{1}{2}\sigma^2(0,u)W^2(u) - \frac{9}{2}\sigma^4(0,u)\int_0^u W^2(u)du$$

(II-21)

Finally we can solve eqns. (II-11) and (II-21) together to obtain ln*F*(0,*T*) as a stochastic process with the mean and volatility

$$< \ln F_{1St}(0,T) >= \mu(1-e^{-\alpha T}) + e^{-\alpha T}\{\ln S(0) - \frac{1}{2\alpha}\sigma_1^2[1-(1+\alpha T)e^{-\alpha T}]$$

$$-\frac{\sigma_1^4}{6\alpha^2}[1-(1+3\alpha T+\frac{1}{2}(3\alpha T)^2)e^{-3\alpha T}]\} \qquad \text{(II-22)}$$

$$Var[\ln F_{1St}(0,T)] = 4\alpha^2\sigma_1^6 e^{-2\alpha T}\int_0^T e^{-2\alpha(u+z)}du\int_0^T dz\int_0^u < W(\tau)d\tau\int_0^z W(t) > dt$$

$$= \frac{\sigma_1^6}{12\alpha^2} T e^{-4\alpha T}\left\{-3 + 2\alpha^2 T^2 + e^{-2\alpha T}(3 + 6\alpha T + 4\alpha^2 T^2)\right\} \tag{II-23}$$

We will compare our result with the conventional case

$$\mu = \text{const} \quad \text{and} \quad \Sigma(t,T) = \sigma_1 e^{-\alpha(T-t)}$$

with

$$\ln F_{1R}(0,T) = \mu(1 - e^{-\alpha T}) + e^{-\alpha T}\ln S(0) - \frac{\sigma_1^2}{4\alpha}(1 - e^{-2\alpha T})^2$$

for specific values of the parameters involved.

By looking at (II-22) and (II-24) we see that the ***means*** differ by the terms proportional to some power of $\sigma_1$.

$$\sigma(t) = \frac{\sigma_1 e^{-\alpha t}}{1 + \sigma_1 e^{-\alpha t}W(t)} \approx \sigma_1 e^{-\alpha t}[1 - \sigma_1 e^{-\alpha t}W(t)] \approx \sigma_1 e^{-\alpha t} \tag{II-24}$$

Underline again that the volatility of the spot price process *S*(*t*) is defined as $\sigma(t) = \Sigma(t,t)$ therefore

Or $\approx \sigma_0 / \left(1 + \sigma_0 W(t)\right) \quad (\alpha T >> 1)$ in the three-parameter model i.e. the spot price volatility falls down with time that looks plausible for the mean reverting process.

## *Three parameter model*

Consider

$$\Sigma_t(0,t) = -\alpha(\Sigma(0,t) + \sigma_0), \quad \text{i.e.} \quad \sigma(0,t) = \sigma_1 e^{-\alpha t} + \sigma_0 \tag{II.25}$$

In this case instead of

$$f(u)=\alpha W(u)\sigma^{2}(0,t)[1-2\sigma(0,t)W(u)]$$

we have

$$f_1(u)=\alpha\sigma_0+\alpha\sigma(0,t)W(u)[(\sigma(0,t)-2\sigma_0)-2\sigma(0,t)(\sigma(0,t)-\frac{3}{2}\sigma_0)W(u)] \qquad \text{(II-26)}$$

and

$$\Phi_1(t)=\alpha\mu-\alpha\frac{\sigma(0,t)}{2}\int_0^t\{(\sigma-2\sigma_0)-2\sigma(2\sigma-3\sigma_0)W(u)+3\sigma^2(3\sigma-4\sigma_0)W^2(u)\}du$$

Then

$$\ln[F(0,T)]=e^{-\alpha T}\left\{\ln S(0)+\int_0^T e^{\alpha u}\Phi_1(u)du\right\}=e^{-\alpha T}\ln S(0)+Int_1$$

And up to the lowest order terms wrt $W(t)$ we obtain

$$\begin{aligned}<\ln[F_{2St}(0,T)]>=e^{-\alpha T}\ln S(0)+\mu(1-e^{-\alpha T})+\frac{1}{2\alpha}\{(\sigma_1^2-\sigma_0^2)e^{-\alpha T}\\ -[\sigma_1^2(1+\alpha T)e^{-2\alpha T}-\sigma_0^2(1-\alpha T)]\}\end{aligned} \qquad \text{(II-27)}$$

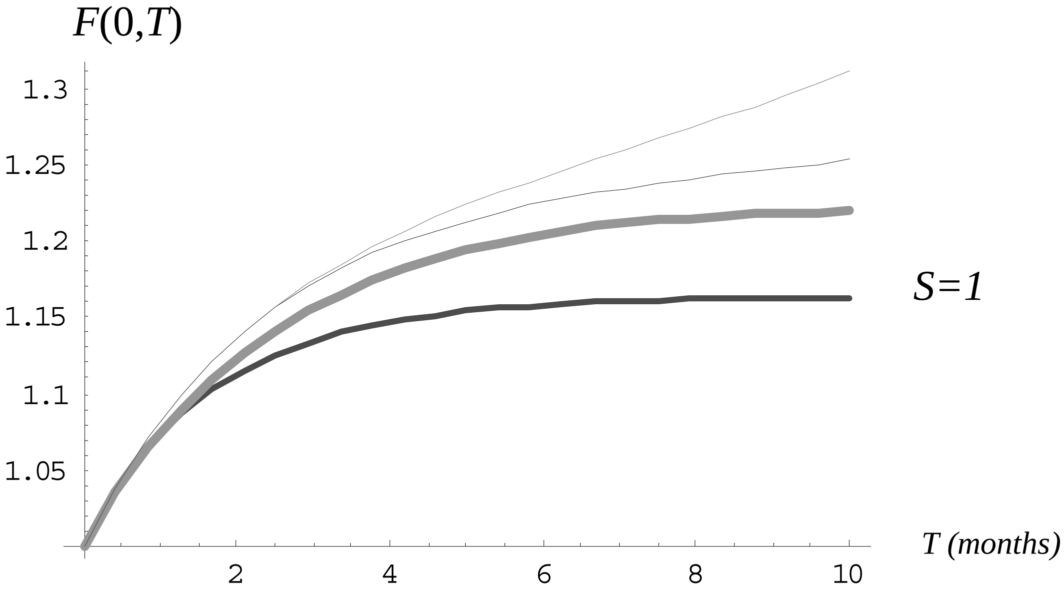
$$Var[\ln F_{2St}(0,T)]=\frac{\sigma_0^2}{2\alpha}\{e^{-2\alpha T}(4e^{\alpha T}-1)+(2\alpha T-3)\} \qquad \text{(II-28)}$$

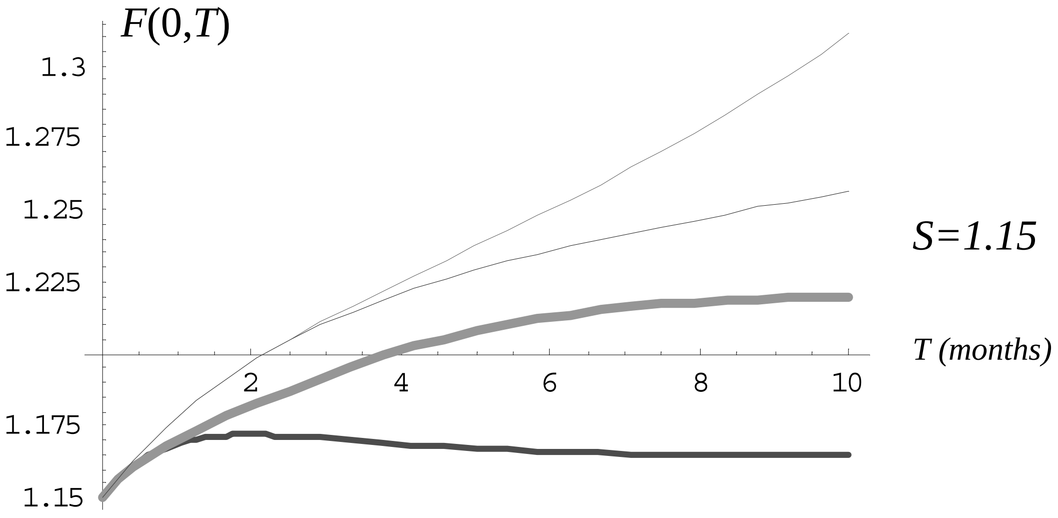
For the regular model (I-8a) we have (1-9a) and (1-9b)

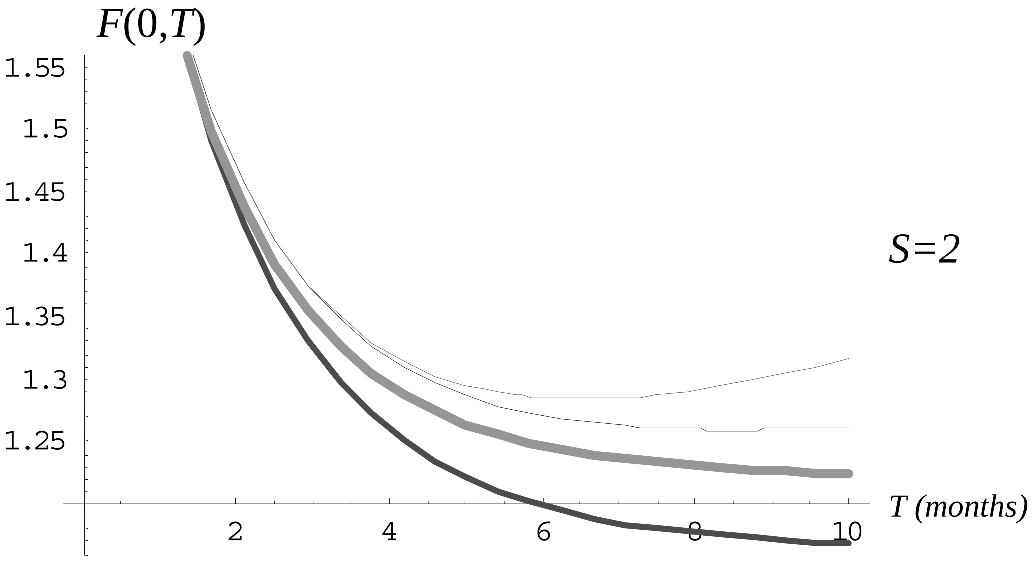
$$<\ln F_{2R}(0,T)>=e^{-\alpha T}S(0)+\mu(1-e^{-\alpha T})-\frac{1}{4\alpha}\left[a^2(1-e^{-\alpha T})^2-2b^2(\alpha T-1+e^{-\alpha T})\right]$$

$$Var\left[\ln F_{2R}(0,T)\right] = \frac{b^2}{2\alpha}\left[2e^{-\alpha T}(1-\alpha T+\alpha^2 T^2) - e^{-2\alpha T}(2+\alpha^2 T^2)\right]$$

The term structures of all these models are given at the pictures for specific values of the parameters:
$\alpha = 0.5$, $\mu = 0.2$, $\sigma_0$, $\sigma_1$ and various $S$ .

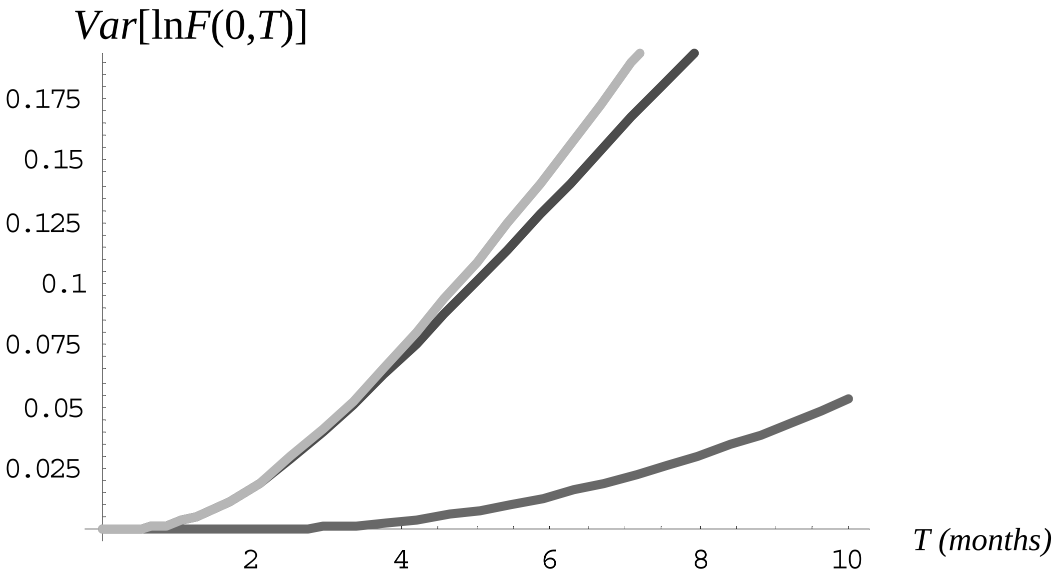






with the lower curve standing for the $F_{1R}(0,T)$, the second from below for $F_{1St}(0,T)$, the third for $F_{2R}(0,T)$ and the fourth for $F_{2St}(0,T)$.
In all calculations we put $\alpha = 0.5, \quad \mu = 0.2, \quad \sigma_1 = 0.14, \quad \sigma_0 = 0.17.$ what is very close to the values at the NYMEX Crude Oil market.
We see that for short terms (around during two months) all four curves look very close to each other and then they start do disperse about 7-10 % at the end of a year.

For the first exponential model the variances for both regular and stochastic variants are negligible. For the second model plots of the stochastic variance (yellow or the top curve), the "regular" one (red or the midlle curve) and the difference between them (magenta, the bottom curve) are as follows.



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