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SHOULD STOCK MARKET INDEXES TIME VARYING CORRELATIONS BE TAKEN INTO ACCOUNT? A CONDITIONAL VARIANCE MULTIVARIATE APPROACH

RYAN LEMAND

ABSTRACT. The episodes of stock market crises in Europe and the U.S.A. since the year 2000, and the fragility of the international stock markets, have sparked the interest of researchers in understanding and in modeling the markets' rising volatilities in order to prevent against crises. Portfolio managers typically rely on estimates of correlations between returns on the financial instruments in the portfolio and on the volatility of those returns. This task is relatively simple if the correlations and volatilities do not change over time. But in reality both volatility and stock market indexes' correlations do change over time. In this paper we examine the major stock market indexes' rising volatilities, and we show that time varying correlations should be taken into account when modeling those indexes. We find that all of the indexes that we examine exhibit relatively time varying correlations with the other indexes and we find a strong GARCH effect in all of the examined series.

1. INTRODUCTION

In periods of heightened market volatility, correlations between returns on financial assets tend to increase relative to correlations estimated during periods of normal volatility. The increased correlation of returns during periods of high volatility is often explained as resulting from changes in the underlying relationships that determine returns. Yet, probability theory shows that correlations between asset returns depend on market volatility even if the underlying relationships between returns have not changed; variations in correlations measured over different periods of time may merely be the consequence of variations in realized volatility.

Modern portfolio theory, since the seminal work of Markowitz (1952), has stressed upon the importance of correlations in the portfolio selection process since returns and volatilities are no longer sufficient in order to have a *good* asset allocation with today's stock market structure. Correlations between assets need to be determined precisely in order to perform an optimal allocation. Otherwise, if time varying correlations that increase during periods of high volatility, are falsely disregarded, then the allocation process is biased; therefore, the portfolio will not be diversified enough as correlations increase during high volatility episodes, which might lead to considerable losses. Ang and Bekaert (1999) pointed this out through the home

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bias puzzle which represents the situation when investors diversify far less internationally than what the theory would predict.
In previous papers, we examined several aspects that link international indexes, specifically through the New Technology sector's channel. We found that there is evidence of co-movement between the NASDAQ-100, the IT.CAC and the NEMAX (Lemand, 2003). We also find that that the co-movement between the the NASDAQ-100 and the IT.CAC is unidirectional and we show that there is a contagion effect from the NASDAQ-100 on the IT.CAC (Lemand, 2002). In these papers, we also put into evidence the fact that the co-movement and the contagion effects are of the volatility type.
In this paper, we explore the link between volatility and correlation, which has until recently been overlooked in the econometrics and finance literature. In fact, the statistical link between sampling volatilities and correlations of asset returns has important implications for the evaluation of portfolio risk by market participants and investors as for the supervision of financial firms' risk management practices.
Risk managers sometimes use data from a relatively short interval when calculating correlations and volatilities for use in risk management models. Some estimation methods are based on longer intervals of data, but they apply geometrically declining weights, thereby reducing the effective number of observations employed. Moreover, risk managers generally use conditional correlations that are volatility independent, more specifically, covariance independent. In fact, using covariance unadjusted conditional correlations would lead to heteroskedasticity biased conditional correlations (Forbes and Rigobon, 2000). So, risk managers should not consider the possible effects of high return volatilities without also taking into account the higher correlations between asset returns that would generally accompany the elevated volatility (Ronn, 1995). Supervisors of financial institutions also need to be aware of the link between volatilities and correlations when assessing firms' risk management practices. For example, in evaluating such firms' internal models, supervisors need to keep in mind the difficulties with relying on a relatively short interval of data for information on correlations and the need to form appropriate conditional correlations for stress tests; the short interval data may disregard previous periods of high covariance and vice-versa.
In this paper, we examine five international stock market indexes' returns (NASDAQ comp., S&P500, DAX, CAC40 and Dow Jones ind.av.). We first start by using a VEC model (Vector Error Correction) to filter the data from any linearities they may contain and we examine the cointegration relationships and the impulse response functions for the five indexes. We then model the residuals of the VEC model using a number of multivariate GARCH models and we attempt to show that Engle and Sheppard's (2001) multivariate dynamic correlation GARCH model fits the data best. We finally conclude that there is no contagion[1] effect between the five indexes but only a strong interdependence that should be taken into account when modeling them.
The paper is organized as follows: section II discusses contagion propagation mechanisms that have been discussed in previous empirical work. Section III provides descriptive statistics of the five indexes and some information on their volatilities and correlations. In section IV we use the VEC model and the impulse response

[1] The definition of contagion is treated later on in this paper

functions. In section V we model the residuals using a number multivariate GARCH models. Section VI is the conclusion.

2. Contagion Propagation mechanisms: previous empirical work

Much of the empirical literature define contagion as a significant increase in cross-market linkages after a shock to one country (or group of countries). Cross-market linkages can be measured by a number of different statistics, such as the correlation in asset returns, impulse response functions, the probability of a speculative attack, or the transmission of shock or volatility, although some of the more recent work has used a broader definition. In this paper we use this definition of contagion through the transmission of volatility channel.
The following four different approaches have been used to test and measure contagion and how the shocks are transmitted internationally: analysis of cross market correlation coefficients, GARCH and heteroskedastic changing regime frameworks, cointegration, and probit models.

2.1. **Cross-market correlation coefficients.** The most straightforward tests are the cross-market correlation coefficients that measure the correlation in returns between two markets during a stable period and a turbulent period and then test if the coefficients have increased in the latter period after the shock. If it did increase significantly after the shock then it can be concluded that the transmission mechanism between the two (or more) markets did significantly take place and the contagion occurred. Most of the papers using this approach dealt with the 1987 U.S. stock market crash, for example, King and Wadhwani (1990) test for an increase in cross-market correlations between the U.S., U.K. and Japan and find that correlations increase significantly after the U.S. 1987 crash. This analysis has been further extended to twelve major markets by Lee and Kim (1993) who also find evidence of contagion following the 1987 crash. Calvo and Reinhart (1995) test the period after the 1994 Mexican peso crisis for contagion in stock prices and Brady bonds. They find that for many emerging markets, the cross-market correlations increased during the crisis.
We can therefore conclude from the tests based on cross-market correlations that these correlations increase significantly during turbulent periods, which suggests that contagion occurred during the period under investigation.

2.2. **GARCH and heteroskedastic changing regime frameworks.** Chou *et. al.* test for contagion using a GARCH framework to estimate the variance-covariance transmission mechanism across markets after the 1987 U.S. stock market crash. They conclude that there are significant spill-overs across markets after the 1987 crash and they find that contagion does not occur evenly across countries, but that it is relatively stable over time. Edwards (1998) uses an augmented GARCH model to examine the propagation across bond markets after the Mexican peso crisis, with a focus on how capital controls affect the transmission of shocks. He finds evidence of significant spill-overs from Mexico to Argentina, but not from Mexico to Chile. So he works on the volatility channel of contagion, which means that volatility is transmitted from one country to another (or more than one country).
Engle and Sheppard (2001) developed the theoretical and empirical properties of a new class of multivariate GARCH models capable of estimating large time-varying

covariance matrices, Dynamic Conditional Correlation Multivariate GARCH (dcc-MVGARCH). This model is a continuity of the constant correlation multivariate GARCH models first introduced by Bollerslev, Engle and Wooldridge (1988) and an elaboration of Engle's (2001) dcc-MVGARCH. Even though this model was not introduced in order to calculate the contagion effect, it can be a very useful tool to calculate dynamic conditional correlations based on the covariance matrix for a large number of series.
Edwards and Susmel (2001) combine ARCH models with regime changing models in order to estimate a bivariate switching ARCH model (SWARCH). They use weekly stock market data for Latin American countries to analyze the behavior of volatility over time. They show that periods of high volatility are correlated across countries and reach the conclusion of a certain level of contagion among those countries. Even though they compute a dynamic correlation coefficient (dcc) using the SWARCH model, their analysis does not rely on the dcc.

2.3. **VAR models and Cointegration.** When using the cointegration approach, changes in the long run relationship between markets is examined instead of the more interesting short-run changes after a shock. In this approach, changes in the cointegrating vector between stock markets are tested instead of in the variance-covariance matrix. Seven O.E.C.D. countries, from 1960 to 1990, are considered in Longin and Slonik (1995), and the average correlations in stock market returns between the U.S. and other countries are reported to have risen over this period. If this type of tests show that the cointegrating relationship increased over time (in case of a shock), this could be a permanent shift in cross-market linkages instead of contagion; this is why this approach has been criticized since it only detects permanent shifts in market linkages and not short lived ones that are not permanent. Moreover, by focusing on long time periods, this set of tests could miss brief periods of contagion (such as after the Russian collapse of 1998).

2.4. **Probit models.** This final approach to testing for contagion uses simplifying assumptions and exogenous events to identify a model and directly measure changes in the propagation mechanism. Baig and Goldfajn (1998) study the impact of daily news, as exogenous event, in one country's stock markets on other countries' markets during the 1997-1998 East Asian crisis. They find that a considerable proportion of a country's news, impacts neighboring countries. Eichengreen, Rose and Wyplosz (1996) examine the E.R.M. countries (France, Germany and Holland) in 1992-1993 using probit models to test how a crisis in one country, as an exogenous event, affects the probability of a crisis occurring in other countries. They find that the probability of a country suffering a speculative attack increases when another country in the E.R.M. is under attack.

3. Descriptive Statistics, Correlations and Volatilities of the Indexes

We work on daily data of five international indexes: NASDAQ Composite, Standard and Poor 500, Dow Jones Industrial Average, German DAX and CAC40. The sample in question covers the period from the 4^{th} of January 1988 till the 31^{st} of January 2003.

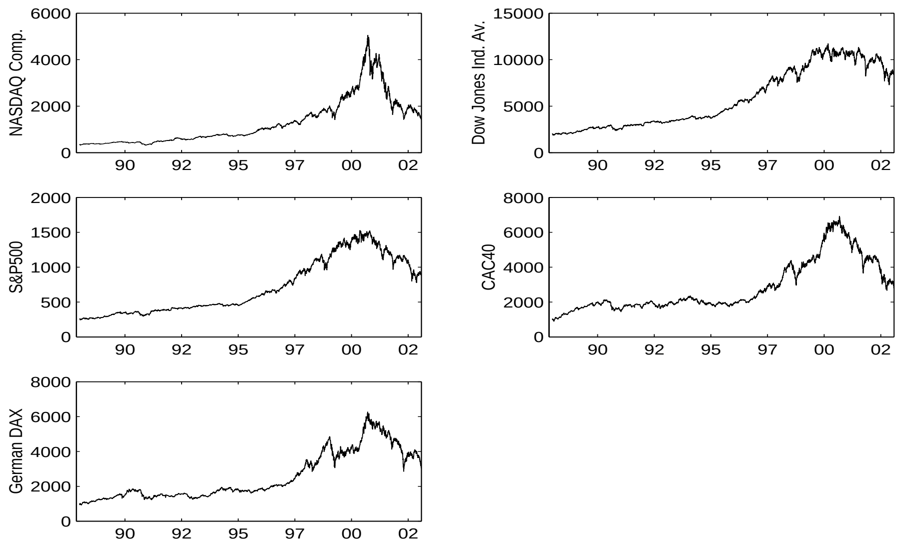

FIGURE 1. The graphs of the five indexes

3.1. **Descriptive Statistics of the Data.** Figure 1 shows the graphs of the five indexes in question.

We use weekly stock returns of the five indexes and the graphs of the five series in question are shown in figure 2.[2].

It can be clearly seen on the graphs of the indexes and the indexes's weekly returns that after the year 1997, the volatilities of the indexes grew quite rapidly with respect to the precedent period. Table 1 provides a summary of the indexes's returns descriptive statistics.

TABLE 1. Descriptive statistics of the five indexes' weekly stock returns

	NASDAQ	DJ	S&P	CAC40	DAX
Mean	0.00	0.00	0.00	0.00	0.00
Median	0.00	0.00	0.00	0.00	0.00
Maximum	0.18	0.10	0.12	0.14	0.13
Minimum	-0.29	-0.14	-0.13	-0.15	-0.17
Std. Dev.	0.03	0.02	0.02	0.03	0.03
Skewness	-0.80	-0.43	-0.36	-0.26	-0.55
Kurtosis	8.27	5.56	5.46	4.69	5.11
Jarque-Bera	4776.81	1142.01	1032.36	491.05	887.19
Probability	0	0	0	0	0
Sum Sq. Dev.	4.14	1.64	1.92	3.44	3.12
Observations	3775	3775	3775	3775	3775

[2]We use weekly stock returns since they are less noisy than daily stock returns and allow for us to keep more information in the series than the first order differencing.

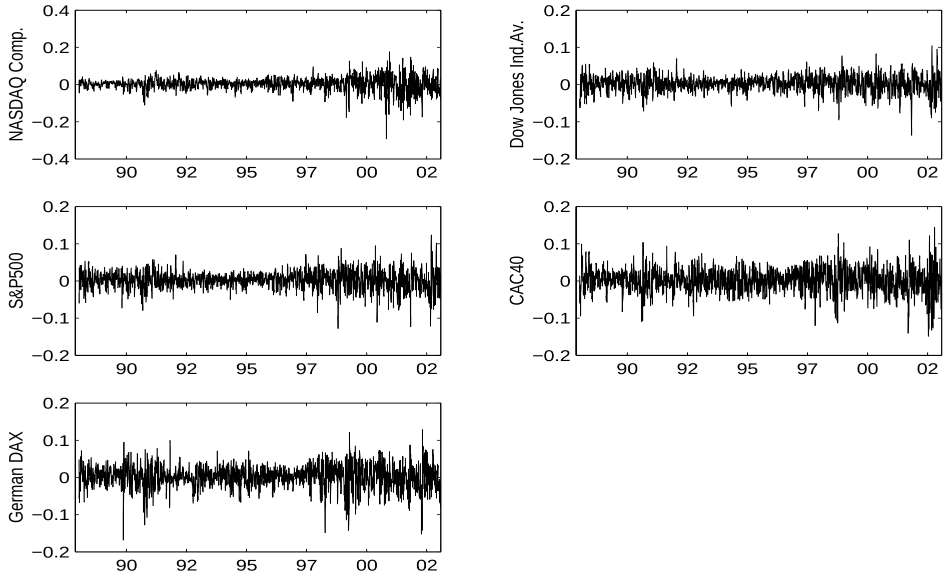

FIGURE 2. The graphs of the five indexes' weekly returns

The average weekly return is negative for the five indexes. Standard deviations reveal that the five indexes' returns have almost the same values. As for the skewness, which is the measure of the distributions' asymmetry of returns, the five indexes' returns have negative skewness values, which suggests that crashes are more likely than booms. As for the kurtosis, which measures the heaviness of tails compared to a measure of three for the normal distribution, we find that the five series exhibit excess kurtosis (larger than 3), therefore their distributions have fatter tails than the normal one. The Jarque-Bera test statistic strongly rejects the normality hypothesis of stock returns for the three indexes. Those preliminary descriptive statistics confirm the widespread results in the financial literature on stock returns: negative skewness and fat tails.

We next consider the presence of return serial correlation. We consider the Ljung-Box statistic. The Ljung-Box (LB) statistic with 36 lags is distributed as a χ^2_{36}. The LB statistic shows significant linear dependencies of returns for the five markets investigated.

Next we consider heteroskedasticity by regressing squared returns on past squared returns (up to 12 lags). The TR^2 Engle statistic, where R^2 is the coefficient of determination, is distributed as a χ^2_{12} under the null hypothesis of homoskedasticity. The Engle statistic takes very large values for each market, and strongly rejects the homoskedasticity null hypothesis, which indicates strong non-linear (second moment) dependencies. We therefore conclude that there is a fair amount of heteroskedasticity in the data.

3.2. **Indexes' Volatilities and Preliminary Evidence on International Correlations.** There has been many recent discussions on the rising volatilities of the

international stock markets in the recent few years. In fact, the year 2000 New Technology sector stock market price correction has rendered investors quite sensitive. The large over-evaluation of many IT (Information and Technology) companies and the uncertainties that arose concerning their accounting methods (especially after the Enron and WorldCom crisis in the U.S.) made investors, whether professional or private, much risk averse. As a consequence we have been witnessing for the past few years large fluctuations of stock returns.
Figure 3 shows the graphs shows the graphs of the 21-day rolling standard deviations (RSD)[3], as a preliminary measure of volatility, for the five indexes weekly returns.

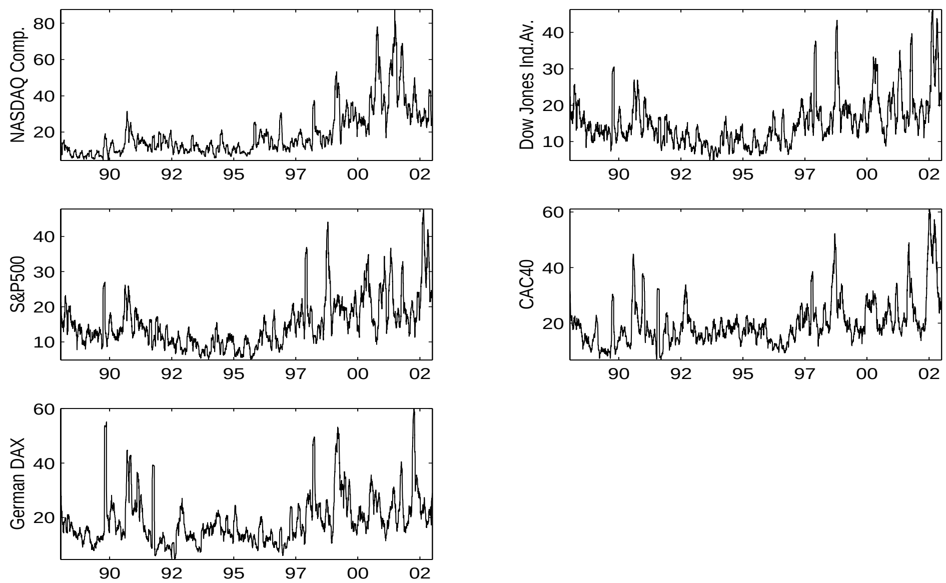

FIGURE 3. The graphs of the five indexes' volatilities using the 21-day rolling standard deviations defined by the RSD (footnote 3)

All indexes returns show a relatively calm period between the years 1992 and mid 1996. Before the year 1992, there were two events that marked the international stock markets: the 1987 stock market crash and the 1990-1991 war to disarm Irak. So at the beginning of the graphs, at the year 1988, the markets were still recovering from the 1987 crash and then shortly afterwards, the Iraki conflict relatively shook the markets. The volatilities take off again with the start of the New Technology speculative bubble around the year 2000 and then the price correction that followed, a price correction that continues till today. The only exception is the NASDAQ which exhibits a low volatility with respect to the other indexes before 1992. In fact, at that period the NASDAQ had a "low profile" in comparison with the Dow Jones Industrial Average and the other international indexes, the New

[3]Rolling standard deviation (RSD) over 21 days: $\sigma(r_t) = [253 \sum_{k=1}^{2} 1(r_{t-k} - \mu)^2/20]^{\frac{1}{2}}$, where μ is the mean of the observations over 21 days and r_t is the stock return (Schwert, 2002).

Technology companies that the NASDAQ represents were not of big interest to the investors.
In order to show the reasons of the volatilities jump in the last few years we examined the number of times in one year when the five indexes had a daily return higher than 3% or lower 3%. Figure 4 shows the graphs of that count for all five indexes.

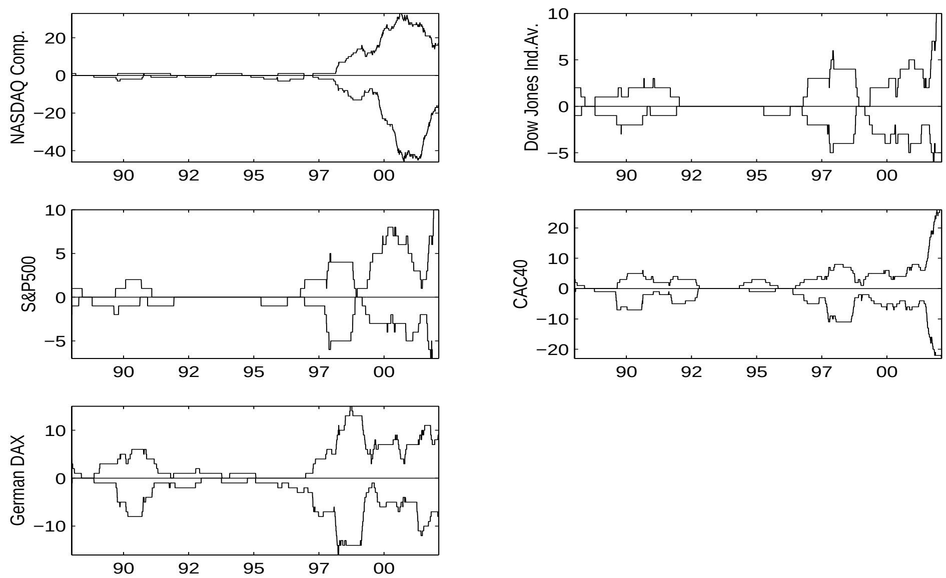

FIGURE 4. Count of daily returns higher than 3% for the five indexes

Looking at figure 4 and comparing it with figure 3, we can clearly see that the volatilities of the indexes follow the pattern shown by the the extreme values (greater or lower than 3% of daily stock returns). We can divide the graphs into three parts or periods: first of all, for the period 1988-1992, the effects of the conflict with Irak and the aftermaths of the 1987 crash are quite clear (except for the NASDAQ), there is a moderate clustering of daily returns that are greater or lower than 3%. As for the second period, 1993-1997, we can clearly see that the indexes we relatively stable having daily returns close to zero. Finally, 1998 till today is a period of great turbulence in comparison with the other two previous periods, the indexes reach upto 80 returns, for the NASDAQ Composite, that are greater or lower than 3%, and that is in only one year (252 trading days).
There are mainly three reasons for this structural change in index returns (discussed in more detail in Lemand, 2003): first of all, investors became more sensitive to the good or bad news that they receive. Second of all, the new instantaneous trading means via the Internet, especially in the case of small private investors, has made it possible for very frequent and easy stock exchange, which also partly explains the high rise in stock market trading volumes. and third of all, investors are affected by the new type of recessions the financial markets have been going through for the past few years (2000-2003). In fact, we witnessed two types of recessions up till

today: the first one is the 1930-1932 great depression type of recessions, where the high deflation rates pushed real interest rates to such high levels that investments dropped severely. The second type of recessions is the one we witnessed in 1957, 1970, 1980-1982 or in 1990-1991 with the rise of inflation and the hardening of monetary policies. So we are living a new type of recessions today, where the excess debt of companies does not obey the traditional theories of economic cycles; prices are stable and monetary policies cannot do any better. One of the reasons that can explain this is the overevaluation of assets, for example, during the latest the fusion and acquisitions that took place in the U.S., the equivalent of $3000 billion of assets have been exchanged. Even though most companies are going through policies of lowering their debt. But still, the ratio of debts to assets is still too high relative to what it was in the 1990s. The economic disequilibrium relative to collective errors of returns on invested capita anticipations is still far from being resolved.

3.3. **Preliminary Evidence on International Correlations.** Table 2 reports unconditional correlation coefficients for the five indexes. Table 2 shows very high

Table 2. Unconditional Correlation Coefficients (in %) for the Five Indexes in Levels

	NASDAQ	DJ	S&P	CAC40	DAX
NASDAQ	100	90	92	95	95
DJ	90	100	99	92	94
S&P	92	99	100	95	94
CAC40	95	93	95	100	94
DAX	95	94	94	94	100

levels for the unconditional correlation coefficients between all five indexes, the highest ones are for the three American indexes. We examine next the unconditional correlation coefficients for the five weekly returns of the five indexes. Table 3 reports the values.

If we examine the unconditional correlation coefficients for the five indexes weekly returns in table 3, we might conclude that most of the indexes are not correlated due to the very low levels of these correlations. But if we examine the conditional correlation coefficients of the five indexes' returns we would conclude otherwise. Figure 5 shows those graphs, we can clearly see that the correlations are not constant and show high levels of fluctuations. Ignoring this variability would seriously affect the choice of portfolios for reasons discussed above.

Next, we look at the correlation coefficients of the 21-day RSDs for the five indexes

Table 3. Unconditional Correlation Coefficients (in %) for the 5 Indexes Weekly Returns

	NASDAQ	DJ	S&P500	CAC40	DAX
NASDAQ	100	-2.90	-0.66	0.82	54.27
DJ	-2.90	100	84.86	7.05	-3.82
S&P500	-0.66	84.86	100	5.00	1.89
CAC40	0.82	7.05	5.00	100	1.31
DAX	54.27	-3.82	-1.89	1.31	100

in order to check the level of volatility correlation of the indexes. Table 4 reports those values and shows fairly high levels of those correlations, which suggests, as we might intuitively think, that the volatilities of the indexes are interlinked.

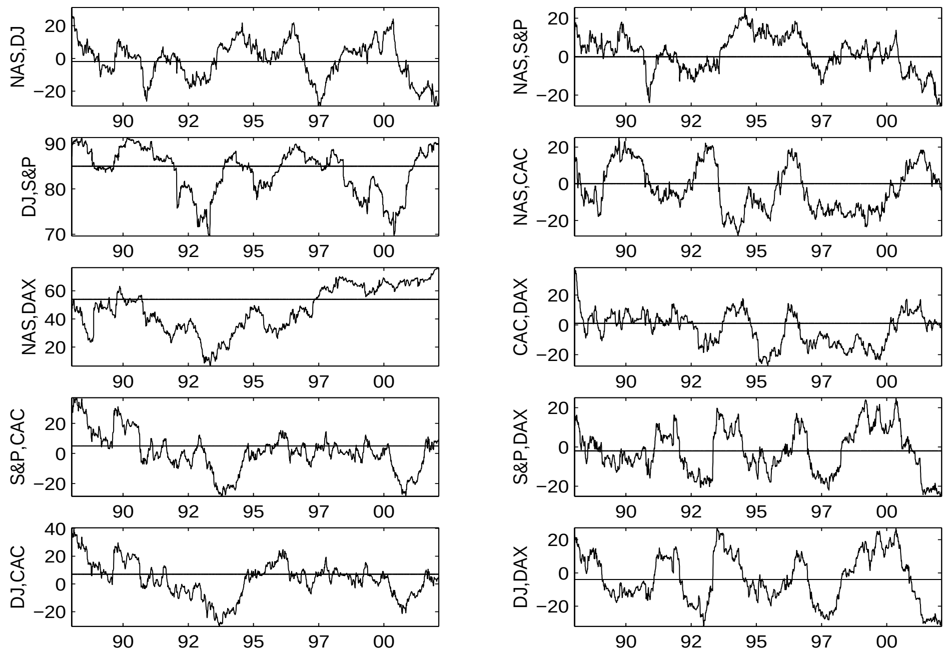

Figure 5. Graphs of the Conditional Correlation Coefficients of the 5 Indexes' Returns (in %)

Table 4. Unconditional Correlation coefficients (in %) of the indexes' 21-day RSD

	NASDAQ	DJ	S&P	CAC40	DAX
NASDAQ	100	37	43	30	51
DJ	37	100	96	51	31
S&P	43	96	100	50	34
CAC40	30	51	50	100	18
DAX	51	31	34	18	100

4. Estimation Techniques

We start by performing an augmented Dickey Fuller test (ADF) in order to identify the presence of a unit root in the data. The result would help us decide on the cointegration order of the three indexes and to construct a Vector Error Correction Model (VECM). The results of the ADF test show that the three series are nonstationary and have a unit root $I(1)$.

A vector error correction model (VECM) is a restricted VAR designed for use with

Table 5. Unrestricted Johansen Cointegration Rank Test for the 5 Series to Determine the Number of Cointegration Equations

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	5 Percent Critical Value	1 Percent Critical Value
None **	0.015418	110.3106	68.52	76.07
At most 1 *	0.007125	51.65491	47.21	54.46
At most 2	0.004739	24.661	29.68	35.65
At most 3	0.001283	6.727331	15.41	20.04
At most 4	0.000498	1.880026	3.76	6.65

non-stationary series that are known to be cointegrated. The VEC has cointegration relations built into the specification so that it restricts the long-run behavior of the endogenous variables to converge to their cointegrating relationships while allowing for short-run adjustment dynamics. The cointegration term is known as the correction term since the deviation from long-run equilibrium is corrected gradually through a series of partial short-run adjustments.

Table 5 shows the results of the Johansen cointegration test to determine the number of cointegration equations (CE) to include in the VEC model. The test reveals the presence of 2 cointegration equations between the indexes.

The VEC model can be represented as follows (Johansen, 1991):

$$\Delta Y_t = \sum_{i=1}^{p-1} D_i \Delta Y_{t-i} + \alpha\beta' Y_{t-1} + \epsilon_t, \tag{4.1}$$

with the cointegration equation defined as a β linear equation between the first variable on one hand and the other two variables on the other hand and so on depending on the number of cointegration equations. Y_t with $t = 1, 2, ..., T$ is the vector of dimension s ($s = 3$ in our case) of the series in question, Δ is the usual difference operator, α and β are matrices of full rank of dimensions $s \times r$ (r is the number of cointegration relations and $0 < r < s$), D_i is a matrix of parameters to be estimated of dimensions $s \times s$, and ϵ_t is a vector of innovations. $\beta' Y_{t-1}$ is the error correction term. It is very important to determine the lag length before estimating the VECM. Therefore we use the Akaike information criteria(AIC), the Schwartz information criteria (SIC) and the likelihood ratio test (LR) to determine the lag length. The main objective of estimating the VECM in this study is to filter the series from any multivariate linearities and to show that the cointegration relationships among them have changed in the past three years.

Next, we carry on with the VEC model estimation using ordinary least squares (OLS), and we use AIC, SC and LR to determine the most convenient lag length. We establish a lag exclusion test after estimation to eliminate unnecessary lags. We carry on a Pairwise Granger Causality/Block Exogeneity Wald Tests after estimation of the VECM to determine if there are any index to be considered as exogenous to the system. The results are reported in table 6. Figure 7 shows the inverse roots of the VEC model that we estimate and shows that they all lie inside the unit circle implying that the VECM is stationary. We can see from table(6) that all the hypotheses are rejected, which means that the three indexes in the three VECM

Table 6. VECM Pairwise Granger Causality/Block Exogeneity Wald Tests for the 5 Indexes

Dep. var.: D(CAC)			
Exclude	Chi-sq	df	Prob.
D(DJ)	90.76185	40	0
D(NAS)	128.1013	40	0
D(SP)	120.9728	40	0
D(DAX)	82.14166	40	0.0001
All	414.1141	160	0
Dep. var.: D(DJ)			
Exclude	Chi-sq	df	Prob.
D(CAC)	199.6803	40	0
D(NAS)	87.23688	40	0
D(SP)	1552.043	40	0
D(DAX)	91.6275	40	0
All	2238.514	160	0
Dep. var.: D(NAS)			
Exclude	Chi-sq	df	Prob.
D(CAC)	81.83241	40	0.0001
D(DJ)	110.7865	40	0
D(SP)	125.3311	40	0
D(DAX)	182.9098	40	0
All	517.9875	160	0
Dep. var.: D(SP)			
Exclude	Chi-sq	df	Prob.
D(CAC)	370.7251	40	0
D(DJ)	115.6008	40	0
D(NAS)	85.2956	40	0
D(DAX)	112.5971	40	0
All	712.3431	160	0
Dep. var.: D(DAX)			
Exclude	Chi-sq	df	Prob.
D(CAC)	67.47911	40	0.0042
D(DJ)	89.21926	40	0
D(NAS)	261.1547	40	0
D(SP)	102.5393	40	0
All	551.7206	160	0

equations are significantly different from zero and that they are endogenous to the system, none should be considered as exogenous.

4.1. **Impulse-Response Functions.** The impulse response function traces the effect of a one standard deviation shock to one of the innovations on current and future values of the endogenous variables. A shock to the i-th variable directly affects the i-th variable, and is also transmitted to all of the endogenous variables through the dynamic structure of the VAR. Since innovations are usually correlated, they have a common component, which cannot be associated with a specific variable. The dynamic analysis of VECMs is usually carried out using the orthogonalized impulse-responses as suggested by Sims (1980). Accordingly, Cholesky decomposition is normally used in the literature where errors are orthogonalized in such a way that the covariance matrix of the resulting innovations is diagonal. We first introduce a shock to the NASDAQ and we analyze the impact within and across the markets over 60 days. We repeat the same in the other four markets namely, Dow Jones Industrial Average, S&P500, CAC40 and Dax. Figure 6 shows the different graphs of the accumulated impulse-response functions.[4]

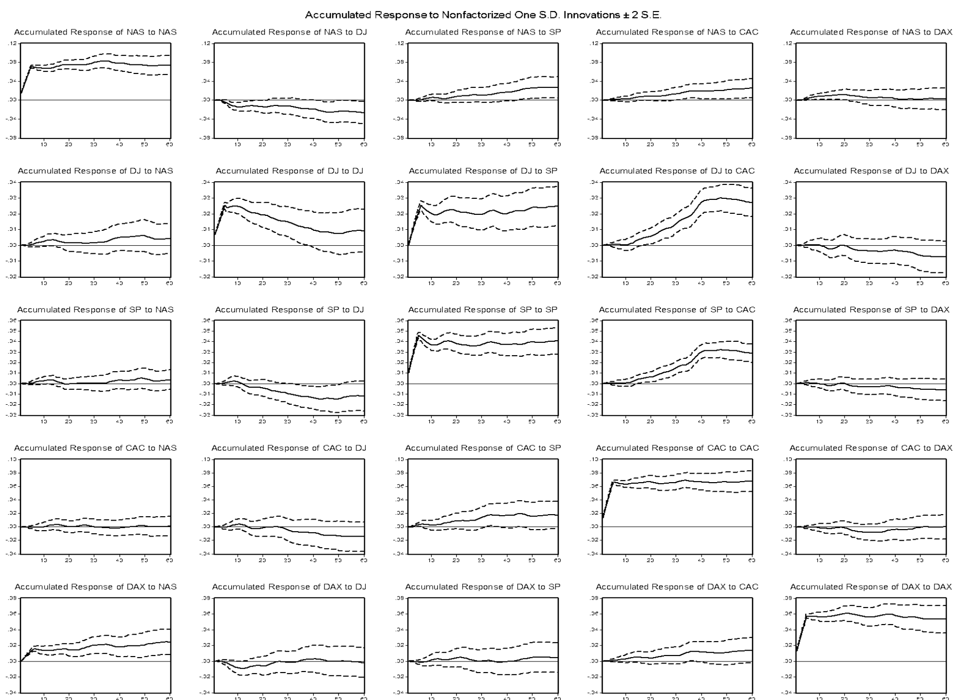

FIGURE 6. Accumulated Impulse-Response Functions Over 60 Days for the 5 Indexes' Returns

The graphs of the impulse-response functions in figure 6 are not very significant, unlike technological indexes as in Lemand (2003). In fact, the dependence of the indexes on each other is not a linear one. The high correlation of the indexes volatilities (table 5) suggests that this dependence or interlinkage is a non-linear one.

Furthermore, if we look at the the two cointegration relations in figure 8, we can clearly see that this longrun relationship has changed after the year 1997, which confirms our hypothesis of a structural change in the trading markets that the five indexes represent.

The RSDs in figure 3 and their unconditional correlations in table 4, along with

[4]The standard errors' bounds are calculated using a 100 iteration Monte Carlo method.

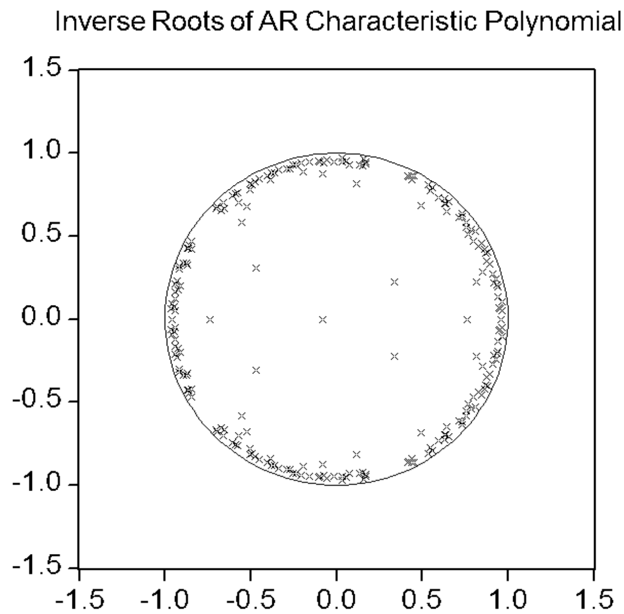

Figure 7. Inverse Roots of the VAR Characteristic Polynomial within the Unit Circle

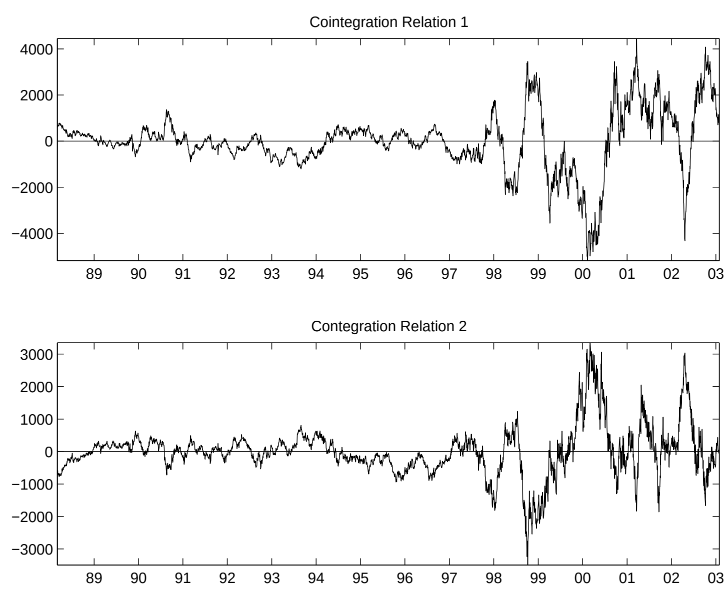

Figure 8. The two Cointegration Relations of the Estimated VECM

the rolling conditional correlation coefficients of the five indexes's returns in figure 5 lead us to consider a conditional variance model with dynamic conditional correlations: Engle and Sheppard's (2001) Dynamic Conditional Correlation Multivariate GARCH (dcc-MVGARCH)[5] model, and we compare it with a number of other multivariate GARCH models.

5. VECM residual analysis using MVGARCH models

We next examine the residuals of the VEC model in order to see if the VEC model has captured the linearity in the data and to see if the residuals are heteroskedastic as it can be expected. We first test the residuals for serial correlation using an LM test for different lags (6, 12 and 18 lags). The results are reported in table 7.

Table 7. VECM Residual Serial Correlation LM Test (H0:no serial correlation at lag order h)

Lags(h)	LM-Stat	Prob
6	28.26	0.31
12	31.65	0.17
24	29.32	0.25

We reject the hypothesis of a presence of serial correlation in the VECM residuals for the three lags, 6, 12 and 24. This means that the VEC model captures all of the linearity that was present in the five indexes. Then, we test the residuals' normality using a multivariate normal test. The results of the test are reported in table (8). The test reveals that the NASDAQ's residuals are symmetrical, whereas the Dow Jones' residuals are positively skewed since the test rejects the hypothesis that its skewness is null. The five indexes's residuals show excess kurtosis, which is typical in the case of financial data. Finally the Jarque-Bera normality test strongly rejects multivariate normality for the five residuals.

We finally examine if the residuals are heteroskedastic. The results of the heteroskedasticity test can be found in table (9).

The joint and the individual heteroskedasticity tests show the presence of heteroskedasticity in the five indexes's residuals, which justifies the use of a multivariate GARCH model.

5.1. **The dcc-MVGARCH Model.**

While univariate GARCH models have succeeded in dealing with a large number of financial issues, multivariate GARCH models (specifically the BEKK family) and time varying correlation multivariate GARCH models have shown their limitations. In fact, large time-varying covariance matrices are needed in portfolio management and optimization. Generally, multivariate GARCH models that estimate more than three series take too much estimation time and often present convergence problems. The dcc-MVGARCH model, introduced by Engle and Sheppard (2001) differ from other specifications in that univariate GARCH models are estimated for each series, and then, using the standardized residuals resulting from the first step, a time varying correlation

[5] In this paper, dcc stands for dynamic conditional correlations and MV for multivariate.

Table 8. VECM normality tests (H0:Residuals are multivariate normal)

Component	Skewness	Chi-sq	df	Prob.
NASDAQ	0.029559	0.543161	1	0.4611
DJ	0.576036	206.2797	1	0
S&P500	-0.65274	264.8751	1	0
CAC40	-0.11993	8.941201	1	0
DAX	-0.60894	230.5217	1	0
Joint		711.1608	5	0
Component	Kurtosis	Chi-sq	df	Prob.
NASDAQ	9.086649	5757.768	1	0
DJ	4.831252	521.1875	1	0
S&P500	6.276983	1668.96	1	0
CAC40	4.309628	266.5592	1	0
DAX	7.572329	3249.171	1	0
Joint		11463.65	5	0
Component	Jarque-Bera	df	Prob.	
NASDAQ	5758.311	2	0	
DJ	727.4672	2	0	
S&P500	1933.835	2	0	
CAC40	275.5004	2	0	
DAX	3479.693	2	0	
Joint	12174.81	10	0	

Table 9. VECM Residual heteroskedasticity Tests: No Cross Terms (only squares)

Joint test:					
Chi-sq	df	Prob.			
12786.64	6750	0			
Individual components:					
Dependent	R-squared	F-test	Prob.	Chi-sq(450)	Prob.
$(ResidNAS)^2$	0.397485	4.807079	0	1482.618	0
$(ResidDJ)^2$	0.313327	3.324893	0	1168.712	0
$(ResidS\&P)^2$	0.288882	2.960103	0	1077.528	0
$(ResidCAC)^2$	0.274981	2.763639	0	1025.678	0
$(ResidDAX)^2$	0.185211	1.656341	0	690.8362	0

matrix is estimated using a simple specification. This approach makes it easy to compute the conditional correlation estimator while preserving the simple interpretation of univariate GARCH models; in fact, this model is extremely time and calculations saving in comparison with the BEKK MVGARCH models (Engle and Kroner, 1995), since it reduces the number of parameters to be estimated (32 parameters for the dcc-MVGARCH against 65 parameters for the BEKK MVGARCH model for five series).
The dcc-MVGARCH model assumes that returns (denoted r_t) from k series are conditionally multivariate normal with zero expected value and covariance matrix

H_t[6], and the returns can be either mean zero or the residuals from a filtered time series (Engle and Sheppard, 2001):

$$\begin{aligned} &\epsilon_t|\Omega_{t-1} \sim N(0, H_t) \\ &Ht \equiv D_t R_t D_t \end{aligned} \tag{5.1}$$

where Ω_t is the matrix of information available till date t, ϵ_t is the series of residuals obtained from the previous VEC model, D_t is the $K \times k$ diagonal matrix of time varying standard deviations from univariate GARCH models with $\sqrt{h_{it}}$ on the i^{th} diagonal, and R_t is the time varying correlation matrix. The log-likelihood function given ϵ_i residuals, $i \in \{1, 2, ..., T\}$ is written as follows:

$$\begin{aligned} L &= -\tfrac{1}{2}\textstyle\sum_{t=1}^{T}(klog(2\pi) + log(|H_t|) + \epsilon_t' H_t^{-1}\epsilon_t) \\ &= -\tfrac{1}{2}\textstyle\sum_{t=1}^{T}(klog(2\pi) + log(|D_t R_t D_t|) + \epsilon_t' D_t^{-1} R_t^{-1} D_t^{-1}\epsilon_t) \\ &= -\tfrac{1}{2}\textstyle\sum_{t=1}^{T}(klog(2\pi) + log(|D_t|) + log(|R_t|) + \eta_t' R_t^{-1}\eta_t) \end{aligned} \tag{5.2}$$

where $\eta_t \sim N(0, R_t)$ are the residuals standardized by their conditional standard deviations. Engle and Sheppard (2001) write the elements of D_t as univariate GARCH models, so that:

$$H_{it} = \omega_i + \sum_{p=1}^{P_i} \alpha_{ip}\epsilon_{it-p}^2 + \sum_{q=1}^{Q_i} \beta_{iq} h_{it-p} \tag{5.3}$$

with H_{it} being the usual GARCH conditional variance (as in Bollerslev, 1986) and for $i = 1, 2, ..., k$ with the usual GARCH restrictions for non-negativity and stationarity being imposed, such as non-negativity of variances and

$$\sum_{p=1}^{P_i} \alpha_{ip} + \sum_{q=1}^{Q_i} \beta_{iq} h_{it-p} < 1. \tag{5.4}$$

The proposed dynamic correlation structure is given by:

[6]The assumption of multivariate normality is not required for consistency and asymptotic normality of the estimated parameters. When the returns have non-Gaussian innovations, the dcc estimator can be interpreted as a quasi-maximum likelihood estimator.

Table 10. Estimation Results and Comparison of the 6 MV-GARCH Specifications

Specification	Loglikel.	No of param.	AIC	SC
Full BEKK	60520	65	60455	60403.92
Scal. BEKK	60364	17	60347	60333.64
Diag. BEKK	60405	25	60380	60360.35
Full T BEKK	60849	66	60783	60731.13
Scal. T BEKK	60763	18	60745	60730.85
Diag. T BEKK	60803	26	60777	60756.57
dcc-MVGARCH	60525	22	60503	60485.71

(5.5)

$$Q_t = (1 - \sum_{m=1}^{M} \alpha_m - \sum_{n=1}^{N} \beta_n)\bar{Q} + \sum_{m=1}^{M} \alpha_m(\eta_t - m\eta'_{t-m}) + \sum_{n=1}^{N} \beta_n Q_{t-n}$$

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1}$$

where α and β are considered as weights, $\bar{Q}$ is the unconditional covariance of the standardized residuals resulting from the first stage estimation and the Q_t^* is a diagonal matrix composed of the square root of the diagonal elements, q_{ij}, of Q_t, and M and N are the dcc lags specified by the researcher. The typical element of R_t will be of the form $\rho_{ijt} = \frac{q_{ijt}}{\sqrt{q_{ii}q_{jj}}}$.

5.2. **Estimation Results for the dcc-MVGARCH Model.** We estimate six multivariate GARCH specifications for the five indexes' residuals: Full BEKK, diagonal BEKK and scalar BEKK with normally distributed and Student-T distributed errors each, in addition to the dcc-MVGARCH specification. Table 10 reports the likelihoods for the seven specifications and the likelihood ratio test and the Akaike (AIC) and Schwartz (SC) criteria in order to choose the best specification[7].

According to the AIC and SC criteria the normally distributed BEKK specifications are preferred over the Student-T distributed ones. Table 11 shows the estimated αs and βs (the GARCH parameters) for the fives indexes' residuals using the dcc-MVGARCH specification.[8]

We examine next the conditional covariances estimated using the dcc-MVGARCH (figure 9). Clearly, the graphs of the time varying variances and covariances show a high volatility after the year 1997, which corresponds to our hypothesis of a structural change in the indexes volatilities following the formation and the explosion of the IT speculative bubble. This rise in volatility corresponds also to the rise in the number of times the indexes' returns were greater or lower than 3%. We can also see on the graphs of figure 11 that there is a small amount of fluctuations around

[7]The second column indicates the maximum value of the log-likelihood function. AIC is calculated as L-k, where k is the number of parameters in column 3. Schwartz is calculated as L-(k/2).ln(T), where L is the value of the loglikelihood and T is the size of the sample.

[8]The numbers in parentheses are the standard errors.

Table 11. Estimated parameters for the dcc-MVGARCH

	Alpha	Beta
NASDAQ	0.12	0.87
	(2.60E-06)	(0.04)
DJ	0.088	0.64
	(1.30E-05)	(0.06)
S&P500	0.08	0.89
	(1.50E-05)	(0.05)
CAC40	0.09	0.86
	(5.30E-06)	(0.06)
DAX	0.01	0.98
	(1.00E-05)	(0.03)

the year 1990, which is due to the Iraki conflict at that time. The period that goes from 1992 till 1997 is a period of calm relative to the period after 1997. We can also note the high covariance of the Dow Jones and the S&P500 when compared with the covariances of the other indexes. Figure 10 shows the graphs of dynamic

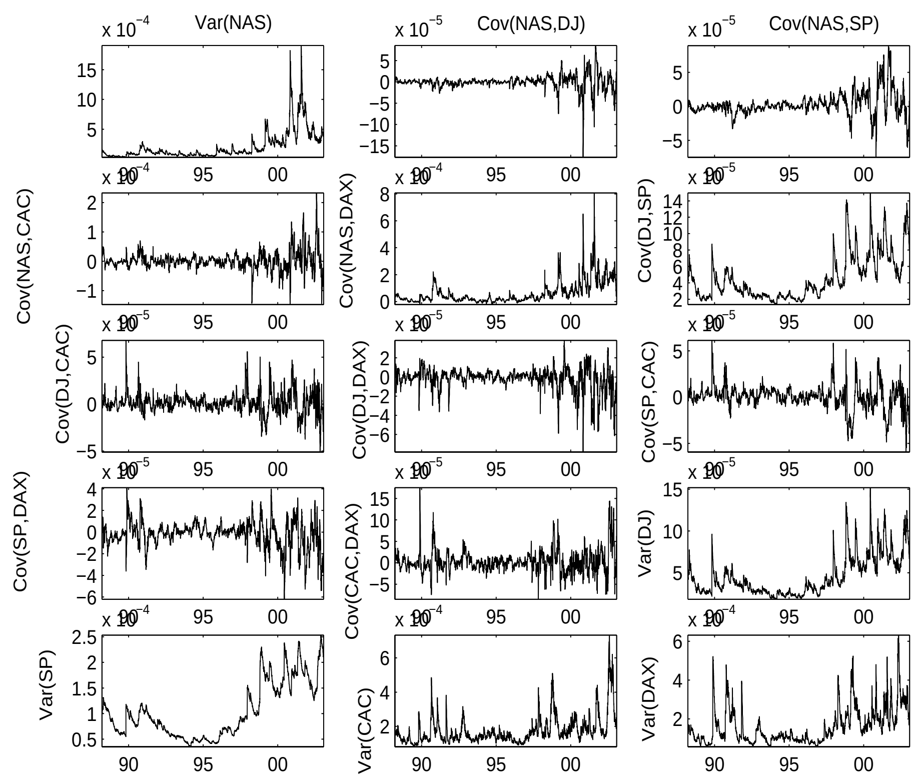

Figure 9. The Variance-Covariance Graphs from the dcc-MVGARCH's Estimation

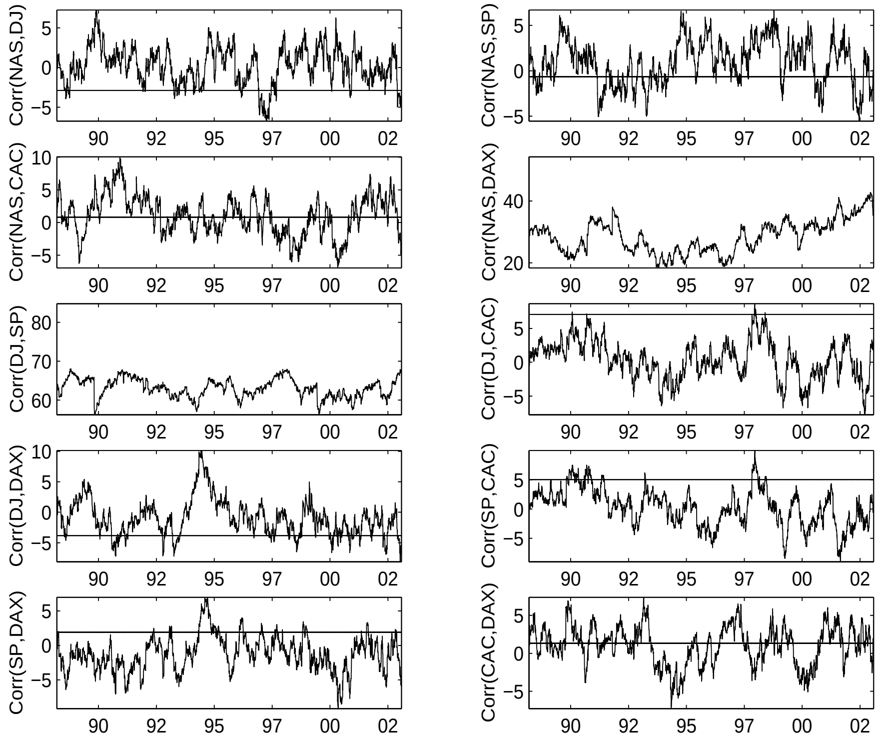

Figure 10. Dynamic Conditional Correlations Graphs from the dcc-MVGARCH's Estimation

conditional correlations estimated with the dcc-MVGARCH. We notice that for the indexes that have a negative unconditional correlation coefficient, the dynamic one has most of its values above the unconditional one. Whereas in the case of positive dynamic conditional correlations, the unconditional correlations are above the conditional ones.

6. Conclusion

In this paper, we model the NASDAQ, Dow Jones Industrial Average, S&P500, CAC40 and German DAX's weekly returns using a linear VEC model and then model the VEC's residuals using a number of MVGARCH specifications, including Engle and Sheppard's (2001) dcc-MVGARCH. We find that the indexes returns' volatilities have considerably increased in the past few years and we find evidence that the unconditional correlations of these returns are not accurate estimators of the return's correlations. In fact portfolio managers should take into consideration the dynamic correlations that exist between the indexes' returns.

This paper can be extended in order to examine the time varying covariances and correlations of stock returns constituting typical portfolios that exist on the market.

The performance difference between considering constant covariances and correlations on one hand, and dynamic ones on the other hand can be studied in order to minimize investment risk in case of crisises.

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Appendix

The intuitive link between volatility and correlation can be derived formally. Boyer *et al* (1999) provide the following theorem.

Theorem. Consider a pair of i.i.d. bivariate normal random variables x and y with standard deviations σ_x and σ_y, respectively, and covariance σ_{xy}. Let $\rho = (\sigma_{xy}/(\sigma_x\sigma_y))$ denote the unconditional correlation between x and y. The correlation between x and y conditional on an event $x \in A$, for any $A \subset \mathbb{R}$ with $0 < Prob(A) < 1$, is given by:

$$\rho_A = \rho\left[\rho^2 + (1-\rho^2)\frac{\sigma_x^2}{var(x|x \in A)}\right] \tag{6.1}$$

Proof.[9] Let u and v be two independent standard normal variables. Now construct two bivariate normal random variables x and y with means μ_x and μ_y, respectively, standard deviations σ_x and σ_y, respectively, and correlation coefficient ρ:

$$\begin{aligned} x &= \mu_x + \sigma_x u \\ y &= \mu_y + \rho\sigma_y u + \sqrt{1-\rho^2}\sigma_y v \end{aligned} \tag{6.2}$$

Consider an event $x \in A$, for any $A \subset \mathbb{R}$ with $0 < Prob(A) < 1$. By definition, the conditional correlation coefficient between x and y , ρ_A, is given by:

$$\rho_A = \frac{cov(x, y|x \in A)}{\sqrt{var(x|x \in A)}\sqrt{var(y|x \in A)}} \tag{6.3}$$

By substituting for u in the second equation of (5.2) using the first equation in (5.2), then substituting the resulting expression for y into (5.3), and using the fact that x and v are independent by construction, one can write this as:

$$\rho_A = \frac{(\rho\sigma_y/\sigma_x)var(x|x \in A)}{\sqrt{var(x|x \in A)}\sqrt{(\rho^2\sigma_y^2/\sigma_x^2)var(x|x \in A) + (1-\rho^2)\sigma_y^2}} \tag{6.4}$$

which can, in turn, be simplified to yield the expression (5.1). Thus, the conditional correlation between x and y is larger (smaller) than ρ in absolute value if

[9] This proof is based on the property of bivariate normal variables that each component can be expressed as the weighted average of the other and of an independent variable that is also normally distributed.

the conditional variance x given $x \in A$ is larger (smaller) than the unconditional variance x.