Levy-stable distributions revisited: tail index > 2 does not exclude the Levy-stable regime

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Abstract: Power-law tail behavior and the summation scheme of Levy-stable distributions is the basis for their frequent use as models when fat tails above a Gaussian distribution are observed. However, recent studies suggest that financial asset returns exhibit tail exponents well above the Levy-stable regime ($0 < \alpha \leq 2$). In this paper we illustrate that widely used tail index estimates (log-log linear regression and Hill) can give exponents well above the asymptotic limit for α close to 2, resulting in overestimation of the tail exponent in finite samples. The reported value of the tail exponent α around 3 may very well indicate a Levy-stable distribution with $\alpha \approx 1.8$.

Keywords: Levy-stable distribution, Tail exponent, Hill estimator, Econophysics

1 Introduction

Levy-stable laws are a rich class of probability distributions that allow skewness and fat tails and have many intriguing mathematical properties [1]. They have been proposed as models for many types of physical and economic systems. There are several reasons for using Levy-stable laws to describe complex systems. First of all, in some cases there are solid theoretical reasons for expecting a non-Gaussian Levy-stable model, e.g. reflection off a rotating mirror yields a Cauchy distribution ($\alpha = 1$), hitting times for a Brownian motion yield a Levy distribution ($\alpha = 0.5, \beta = 1$), the gravitational field of stars yields the Holtsmark distribution ($\alpha = 1.5$) [2, 3, 4]. The second reason is the Generalized Central Limit Theorem which states that the only possible non-trivial limit of normalized sums of independent identically distributed terms is Levy-stable [5]. It is argued that some observed quantities are the sum of many small terms – asset prices, noise in communication systems, etc. – and hence a Levy-stable model should be used to describe such systems. The third argument for modeling with Levy-stable distributions is empirical: many large data sets exhibit fat tails (or heavy tails, as they are called in the mathematical literature) and skewness, for a review see [4, 6, 7]. Such data sets are poorly described

by a Gaussian model and usually can be quite well described by a Levy-stable distribution.

Recently, in a series of economic and econophysics articles the Levy-stability of returns has been rejected based on the log-log linear regression of the cumulative distribution function or the Hill estimator [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In this paper we show that the cited estimation methods can give exponents well above the asymptotic limit for Levy-stable distributions with α close to 2, which results in overestimation of the tail exponent in finite samples. As a consequence, the reported value of the tail exponent α around 3 may suggest a Levy-stable distribution with $\alpha \approx 1.8$.

2 Levy-stable distributions

Levy-stable laws were introduced by Paul Levy during his investigations of the behavior of sums of independent random variables in the early 1920's [18]. The lack of closed form formulas for probability density functions for all but three Levy-stable distributions (Gaussian, Cauchy and Levy), has been a major drawback to the use of Levy-stable distributions by practitioners. However, now there are reliable computer programs to compute Levy-stable densities, distribution functions and quantiles [19]. With these programs, it is possible to use Levy-stable models in a variety of practical problems.

The Levy-stable distribution requires four parameters to describe: an index of stability (tail index, tail exponent or characteristic exponent) $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \in R$. The tail exponent α determines the rate at which the tails of the distribution taper off, see Fig. 1. When $\alpha = 2$, a Gaussian distribution results. When $\alpha < 2$, the variance is infinite. When $\alpha > 1$, the mean of the distribution exists and is equal to μ. In general, the p–th moment of a Levy-stable random variable is finite if and only if $p < \alpha$. When the skewness parameter β is positive, the distribution is skewed to the right. When it is negative, it is skewed to the left. When $\beta = 0$, the distribution is symmetric about μ. As α approaches 2, β loses its effect and the distribution approaches the Gaussian distribution regardless of β. The last two parameters, σ and μ, are the usual scale and location parameters, i.e. σ determines the width and μ the shift of the mode (the peak) of the distribution.

2.1 Characteristic function representation

Due to the lack of closed form formulas for densities, the Levy-stable distribution can be most conveniently described by its characteristic function $\phi(t)$ – the inverse Fourier transform of the probability density function. However, there are multiple parameterizations for Levy-stable laws and much confusion has been caused by these different representations [20]. The variety of formulas is caused by a combination of historical evolution and the numerous problems that have been analyzed using specialized forms of the Levy-stable distributions. The most popular parameterization

of the characteristic function of $X \sim S_\alpha(\sigma, \beta, \mu)$, i.e. a Levy-stable random variable with parameters α, σ, β and μ, is given by [21, 22]:

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log |t|\} + i\mu t, & \alpha = 1. \end{cases} \quad (1)$$

For numerical purposes, it is often useful [23] to use a different parameterization:

$$\log \phi_0(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 + i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2} [(\sigma |t|)^{1-\alpha} - 1]\} + i\mu_0 t, & \alpha \neq 1, \\ \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log(\sigma |t|)\} + i\mu_0 t, & \alpha = 1. \end{cases} \quad (2)$$

The $S_\alpha^0(\sigma, \beta, \mu_0)$ parameterization is a variant of Zolotariev's [3] (M)-parameterization, with the characteristic function and hence the density and the distribution function jointly continuous in all four parameters. In particular, percentiles and convergence to the power-law tail vary in a continuous way as α and β vary. The location parameters of the two representations are related by $\mu = \mu_0 - \beta\sigma \tan \frac{\pi\alpha}{2}$ for $\alpha \neq 1$ and $\mu = \mu_0 - \beta\sigma \frac{2}{\pi} \log \sigma$ for $\alpha = 1$.

For simplicity, in Section 3 we will analyze only non-skewed ($\beta = 0$) Levy-stable laws. This is not a very restrictive assumption, since most financial asset returns exhibit only slight skewness. For $\beta = 0$ both representations are equivalent, however, in the general case the S^0 representation is preferred.

2.2   Simulation of Levy-stable variables

The complexity of the problem of simulating sequences of Levy-stable random variables results from the fact that there are no analytic expressions for the inverse F^{-1} of the cumulative distribution function. The first breakthrough was made by Kanter [24], who gave a direct method for simulating $S_\alpha(1, 1, 0)$ random variables, for $\alpha < 1$. It turned out that this method could be easily adapted to the general case. Chambers, Mallows and Stuck [25] were the first to give the formulas.

The algorithm for constructing a random variable $X \sim S_\alpha(1, \beta, 0)$, in representation (1), is the following [22]:

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;

- for $\alpha \neq 1$ compute:

$$X = S_{\alpha,\beta} \times \frac{\sin(\alpha(V + B_{\alpha,\beta}))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_{\alpha,\beta}))}{W}\right)^{(1-\alpha)/\alpha}, \quad (3)$$

where

$$B_{\alpha,\beta} = \frac{\arctan(\beta \tan \frac{\pi\alpha}{2})}{\alpha},$$

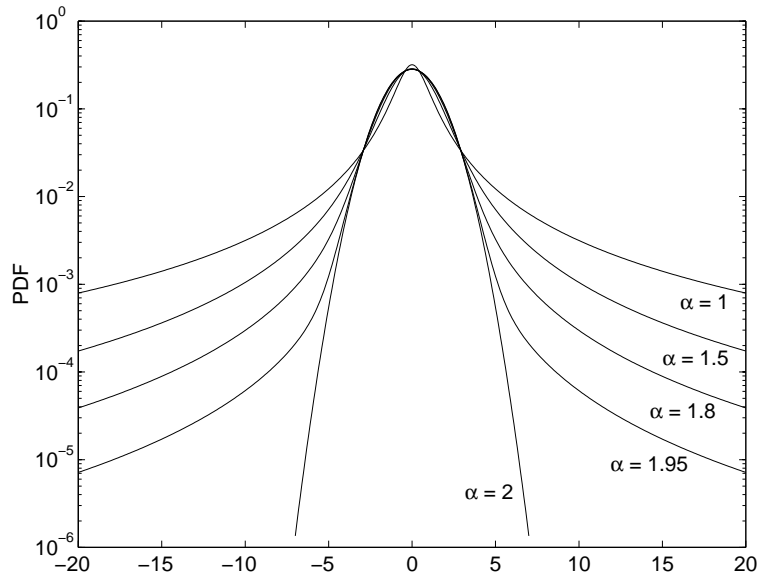$$S_{\alpha,\beta} = \left[1 + \beta^2 \tan^2 \frac{\pi\alpha}{2}\right]^{1/(2\alpha)};$$

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Figure 1: A semilog plot of symmetric ($\beta = \mu = 0$) Levy-stable probability density functions for $\alpha = 2, 1.95, 1.8, 1.5$ and 1. Observe that the Gaussian ($\alpha = 2$) density forms a parabola and is the only Levy-stable density with exponential tails.
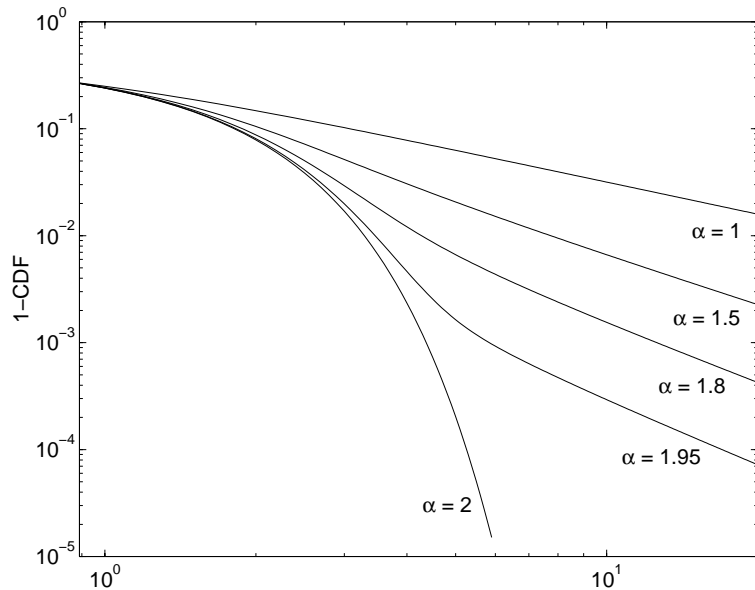

Figure 2: A double logarithmic plot of the right tails of symmetric Levy-stable cumulative distribution functions for $\alpha = 2, 1.95, 1.8, 1.5$ and 1. For $\alpha < 2$, the power tails are clearly visible. Moreover, the smaller the tail index the stronger is the power decay behavior. Recall that the Gaussian tails decay much faster, i.e. exponentially.

- for $\alpha = 1$ compute:

$$X = \frac{2}{\pi} \left[\left(\frac{\pi}{2} + \beta V \right) \tan V - \beta \log \left(\frac{\frac{\pi}{2} W \cos V}{\frac{\pi}{2} + \beta V} \right) \right]. \qquad (4)$$

Given the formulas for simulation of standard Levy-stable random variables, we can easily simulate a Levy-stable random variable for all admissible values of the parameters α, σ, β and μ using the following property: if $X \sim S_\alpha(1, \beta, 0)$ then

$$Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1, \\ \\ \sigma X + \frac{2}{\pi} \beta \sigma \log \sigma + \mu, & \alpha = 1, \end{cases}$$

is $S_\alpha(\sigma, \beta, \mu)$. The presented method is regarded as the fastest and the best one known. It is widely used in many software packages, including S-plus and STABLE [19].

2.3 Tail behavior

Levy [18] has shown that when $\alpha < 2$ the tails of Levy-stable distributions are asymptotically equivalent to a Pareto law. Namely, if $X \sim S_{\alpha < 2}(1, \beta, 0)$ then as $x \to \infty$:

$$P(X > x) = 1 - F(x) \quad \to \quad C_\alpha (1 + \beta) x^{-\alpha},$$

$$\qquad (5)$$

$$P(X < -x) = F(-x) \quad \to \quad C_\alpha (1 - \beta) x^{-\alpha},$$

where

$$C_\alpha = \left(2 \int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin \frac{\pi \alpha}{2}.$$

The convergence to a power-law tail varies for different α's and, as can be seen in Fig. 2, is slower for larger values of the tail index. Moreover, the tails of Levy-stable distribution functions exhibit a crossover from a power decay with exponent $\alpha > 2$ to the true tail with exponent α. This phenomenon is more visible for large α's and will be investigated further in the next Section.

3 Estimation of the tail index

The problem of estimating the tail index (as well as other parameters) is in general severely hampered by the lack of known closed–form density functions for all but a few members of the Levy-stable family. Fortunately, there are numerical methods that have been found useful in practice.

Currently there exist three estimation procedures for estimating Levy-stable law parameters worth recalling. The first one is based on a numerical approximation of the Levy-stable likelihood function. The ML method, as it is called, was originally

developed by DuMouchel [26] and recently optimized by Nolan [27]. It is the slowest of the three but possesses well known asymptotic properties.

The second method uses a regression on the sample characteristic function. It is both fast and accurate (as long as we are dealing with a sample generated by a Levy-stable law). The regression procedure was developed in the early 1980's by Koutrouvelis [28, 29] and recently improved by Kogon and Williams [30].

The last, but not least, is the quantile method of McCulloch [31]. It is the fastest of the three, because it is based on tabulated quantiles of Levy-stable laws. Yet it lacks the universality of the other two, since it is restricted to $\alpha \geq 0.6$.

All presented methods work pretty well assuming that the sample under consideration is indeed Levy-stable. However, if the data comes from a different distribution, these procedures may mislead more then the Hill and direct tail estimation methods. And since there are no formal tests for assessing the Levy-stability of a data set we suggest to first apply the "visual inspection" or non-parametric tests to see whether the empirical densities resemble those of Levy-stable laws.

3.1   Log-log linear regression

The simplest and most straightforward method of estimating the tail index is to plot the right tail of the (empirical) cumulative distribution function (i.e. $1 - F(x)$) on a double logarithmic paper, as in Figs. 2-7 (see also Refs. [13, 14, 15, 16]). The slope of the linear regression for large values of x yields the estimate of the tail index α, through the relation $\alpha = -\text{slope}$.

This method is very sensitive to the sample size and the choice of the number of observations used in the regression. Moreover, the slope around -3 may indicate a non-Levy-stable power-law decay in the tails or the contrary – a Levy-stable distribution with $\alpha \approx 1.8$. To illustrate this, we simulated (using Eq. (3)) samples of size $N = 10^4$ and 10^6 of standard symmetric ($\beta = \mu = 0$, $\sigma = 1$) Levy-stable distributed variables with $\alpha = 1.95$ and 1.8. Next, we plotted the right tails of the empirical distribution functions on a double logarithmic paper. For α close to 2 the true tail behavior (5) is observed only for very large (also for very small, i.e. the negative tail) observations, see Figs. 3 and 5, after a crossover from a temporary power-like decay. Moreover, the obtained estimates still have a slight positive bias: 0.03 for $\alpha = 1.8$ and 0.12 for $\alpha = 1.95$, which suggests that perhaps even larger samples than 10^6 observations should be used.

To test the method for extremely large data sets we also simulated samples of size $N = 10^8$ of standard symmetric Levy-stable distributed variables with $\alpha = 1.95$ and 1.8. As can be seen in Fig. 7 the improvement over one million records samples is not substantial. The tail index estimate (2.02) is closer to the true value of $\alpha = 1.95$, but still outside the Levy-stable regime. Similarly, for $\alpha = 1.8$ the method overestimated the tail exponent and returned $\alpha = 1.82$.

If a typical size data set is used, i.e. 10^4 observations or less as in Refs. [10, 17] and some data sets in Refs. [14, 15, 16], the plot may be quite misleading. An empirical cumulative distribution function of a Levy-stable sample of size 10^4 and $\alpha > 1.5$ does not exhibit the true tail behavior ($x^{-\alpha}$ decay), but a temporary power-

like decay with the slope (more precisely: absolute value of the slope) significantly greater then 2: |slope| = 4.99 for $\alpha = 1.95$ and 2.89 for $\alpha = 1.8$; see Figs. 4 and 6 and compare with Figs. 3 and 5, respectively. Slight differences in the slope of the initial temporary power-like decay (4.91 in Fig. 3 compared to 4.99 in Fig. 4 and 2.95 in Fig. 5 compared to 2.89 in Fig. 6) are caused by inaccurate estimation for large observations in the smaller samples.

Figures 3-7 clearly illustrate that the true tail behavior of Levy-stable laws is visible only for extremely large data sets. In practice, this means that in order to estimate α we must use high-frequency asset returns and restrict ourselves to the most "outlying" observations. Otherwise, inference of the tail index may be strongly misleading and rejection of the Levy-stable regime unfounded. In Figures 3, 5 and 7 we used only the upper 0.5% or less of the records to estimate the true tail exponent. In general, the choice of observations used in the regression is subjective and can yield large estimation errors.

3.2 Hill estimator

Hill [32] proposed a method for estimating the tail index that does not assume a parametric form for the entire distribution function, but focuses only on the tail behavior. The Hill estimator is used to estimate the (Pareto) tail index α, when the upper tail [33] of the distribution is of the form: $1 - F(x) = Cx^{-\alpha}$. If $X_{(1)}, X_{(2)}, ..., X_{(N)}$ is the order statistics, i.e. original sample ordered so that $X_{(1)} \geq X_{(2)} \geq ... \geq X_{(N)}$, drawn from a population with law F then the Hill estimate of α based on the k largest order statistics is:

$$\alpha_{Hill}(k) = \left(\frac{1}{k} \sum_{n=1}^{k} \log \frac{X_{(n)}}{X_{(k+1)}} \right)^{-1}. \tag{6}$$

Unfortunately, it is difficult to choose the right value of k. In practice, $\alpha_{Hill}(k)$ is plotted against k and one looks for a region where the plot levels off to identify the correct order statistic [6, 34]. Algorithms for choosing "optimal" k have been proposed in the literature (see e.g. Refs. [35, 36]), but usually give numbers in the vicinity of the plateau.

To illustrate the performance of the Hill estimator for Levy-stable laws, we simulated (using Eq. (3)) samples of size $N = 10^4, 10^6$ and 10^8 of standard symmetric ($\beta = \mu = 0$, $\sigma = 1$) Levy-stable distributed variables with $\alpha = 1.95, 1.8, 1.5$ and 1. Next, we plotted the Hill statistic $\alpha_{Hill}(k)$ vs. k and compared estimated and true values of α.

Figure 8 presents the results of the analysis for Levy-stable samples of size 10^4. As has been reported in the literature [34, 37, 38], for $\alpha \leq 1.5$ the estimation is within reasonable limits but as α approaches 2 the Hill estimate is well above the Levy-stable regime. In the "extreme" case of $\alpha = 1.95$ there is no single value of k that gives the right value of the tail exponent! For the 1.8-stable sample Hill estimates are close to the true value of α only for $k < 50$, whereas the plateau and "optimal" order statistic [36] are in the range $k \in (200, 400)$ yielding $\alpha_{Hill} \approx 2.85$. Actually the results are very close to those of the log-log linear regression, see Figs.
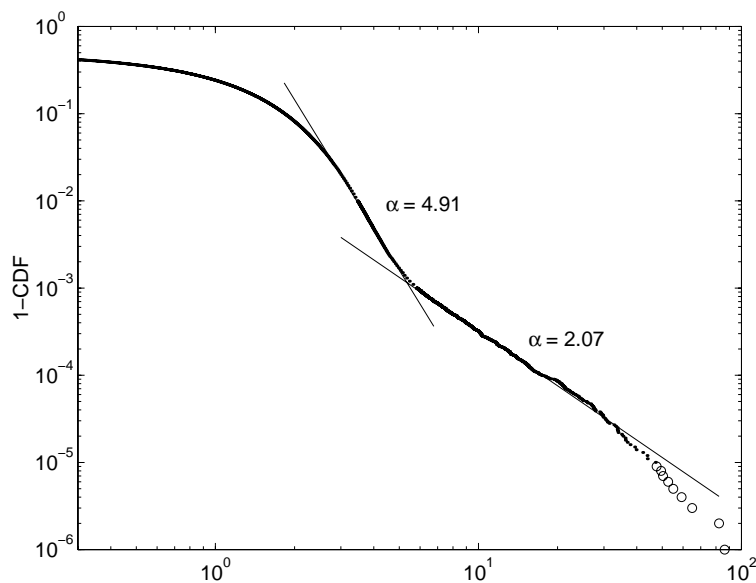
Figure 3: A double logarithmic plot of the right tail of an empirical symmetric 1.95-stable distribution function for sample size $N = 10^6$. Circles represent outliers which were not used in the estimation process. Even the far tail estimate $\alpha = 2.07$ is above the Levy-stable regime.
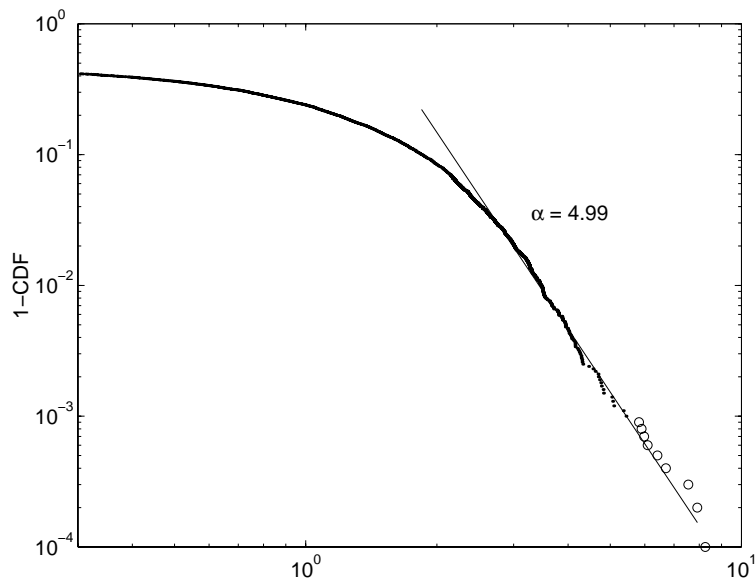

Figure 4: A double logarithmic plot of the right tail of an empirical symmetric 1.95-stable distribution function for sample size $N = 10^4$. Circles represent outliers which were not used in the estimation process. This example shows that inference of the tail exponent from samples of typical size is strongly biased.
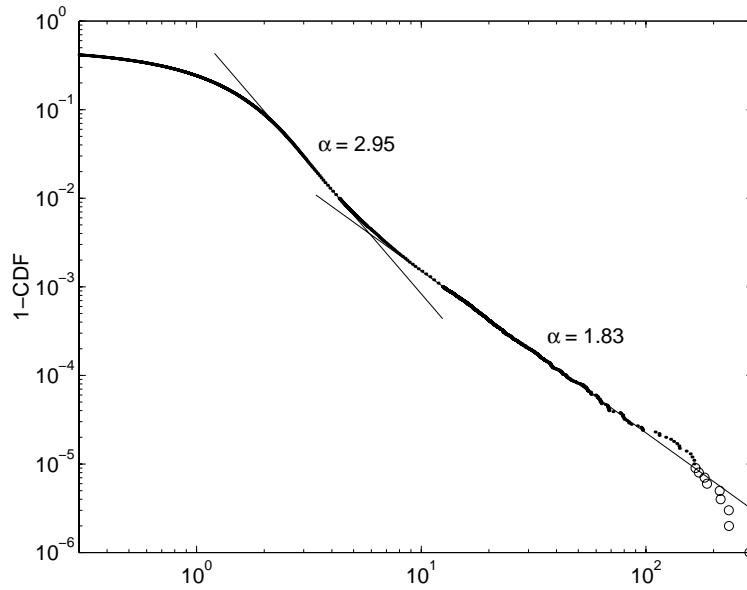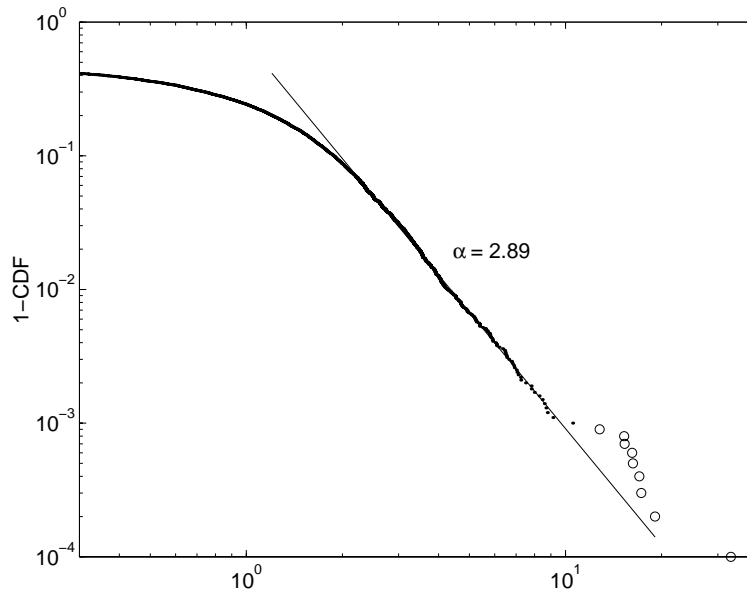
Figure 5: A double logarithmic plot of the right tail of an empirical symmetric 1.8-stable distribution function for sample size $N = 10^6$. Circles represent outliers which were not used in the estimation process. The far tail estimate $\alpha = 1.83$ is slightly above the true value of α.

Figure 6: A double logarithmic plot of the right tail of an empirical symmetric 1.8-stable distribution function for sample size $N = 10^4$. Circles represent outliers which were not used in the estimation process. This example shows that inference of the tail exponent from samples of typical size is strongly biased and the reported value of the tail exponent around 3 may very well indicate a Levy-stable distribution with $\alpha \approx 1.8$.
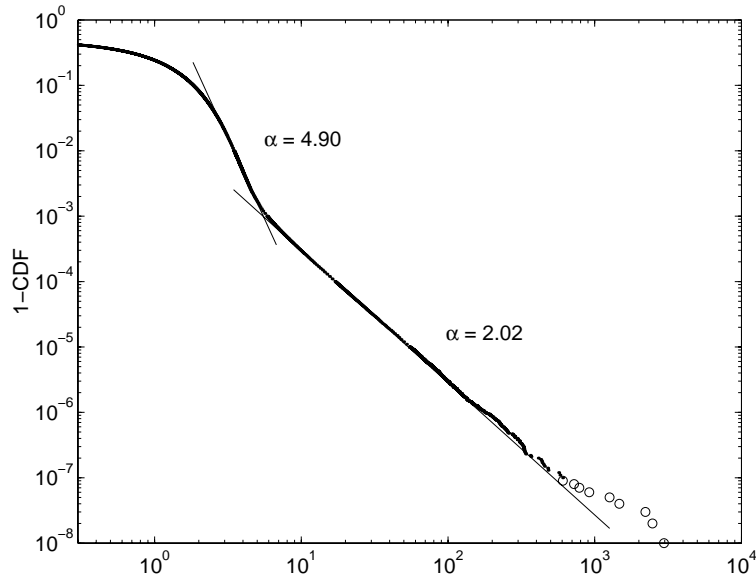
Figure 7: A double logarithmic plot of the right tail of an empirical symmetric 1.95-stable distribution function for sample size $N = 10^8$. Circles represent outliers which were not used in the estimation process. Even the far tail estimate $\alpha = 2.02$ is above the Levy-stable regime.
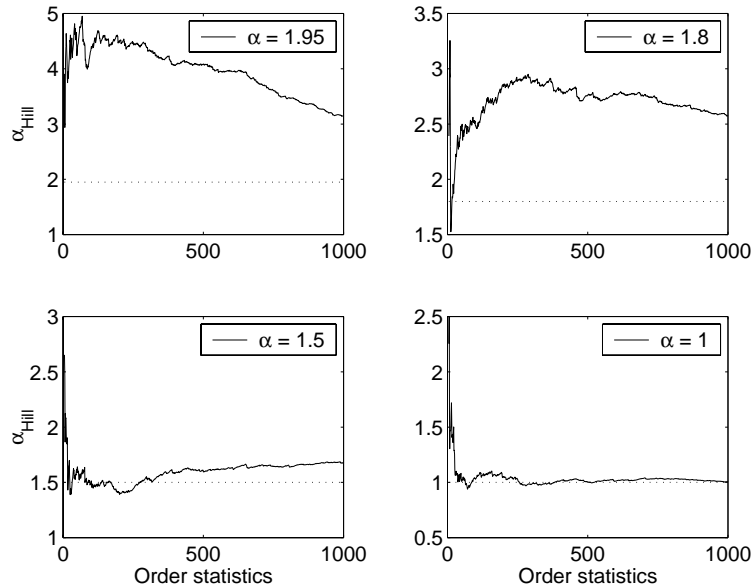

Figure 8: Plots of the Hill statistics α_{Hill} vs. the maximum order statistic k for $1.95, 1.8, 1.5$ and 1-stable samples of size $N = 10^4$. Dashed lines represent the true value of α. For α close to 2, the Hill tail estimator has a large positive bias resulting in overestimation of the tail exponent.
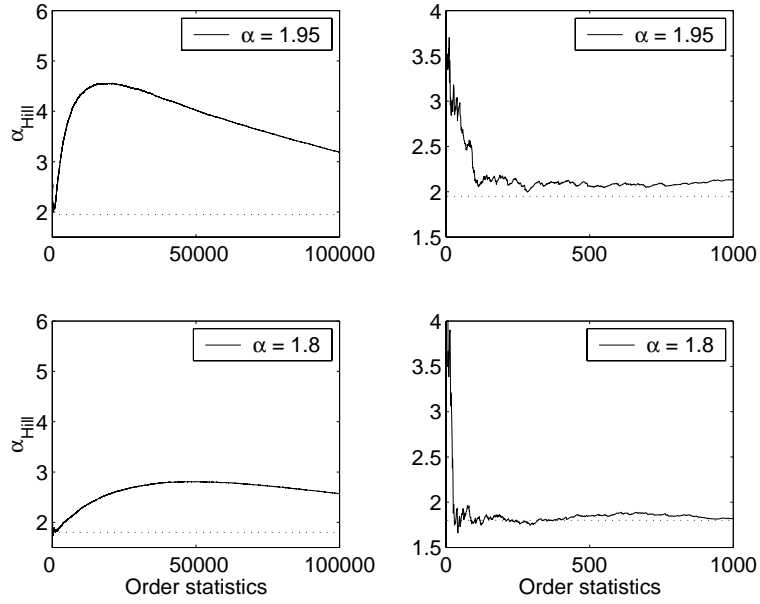
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Figure 9: Plots of the Hill statistics α_{Hill} vs. the maximum order statistic k for 1.95 and 1.8-stable samples of size $N = 10^6$. Dashed lines represent the true value of α. For better visibility, right plots are a magnification of the left plots for small k. For $\alpha = 1.8$ a good estimate is obtained only for $k = 50, ..., 400$ (i.e. for $k < 0.04\%$ of sample size), whereas for $\alpha = 1.95$ the estimate is always above the Levy-stable regime.
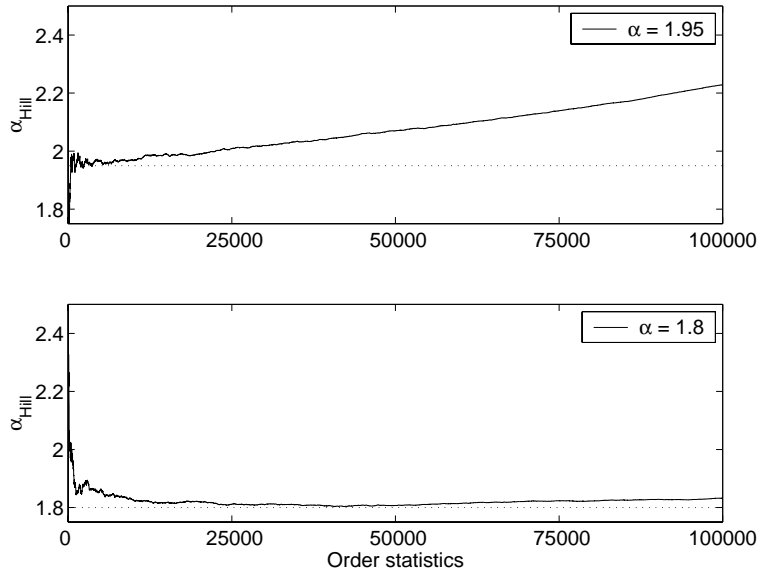

Figure 10: Plots of the Hill statistics α_{Hill} vs. the maximum order statistic k for 1.95 and 1.8-stable samples of size $N = 10^8$. Dashed lines represent the true value of α. For $\alpha = 1.8$ a good estimate is obtained only for $k = 25000, ..., 60000$ (i.e. for $k < 0.06\%$ of sample size), whereas for $\alpha = 1.95$ the estimate is in the Levy-stable regime only for $k < 20000$ (i.e. for $k < 0.02\%$ of sample size).

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4 and 6 and Table 1, since in both methods estimates are obtained from the largest order statistics.

Table 1: Pareto (power-law) tail index α estimates for Levy-stable samples of size 10^4.

	Estimation method	
Simulated α	Log-log regression	Hill
1.95	4.99	$4.4 - 4.6$
1.8	2.89	$2.8 - 2.9$
1.5	—	$1.5 - 1.6$
1.0	—	$1.0 - 1.1$

Figures 9 and 10 present a detailed study of the Hill estimator for samples of size $N = 10^6$ and 10^8. Since for $\alpha \leq 1.5$ the results were satisfactory even for much smaller samples, in this study we restricted ourselves to $\alpha = 1.95$ and 1.8. When looking at the left panels of Fig. 9, i.e. for $k < 10\%$ of the sample size (as in Fig. 8), the flat regions suggest similar values of α_{Hill} as did the Hill plots in Fig. 8. However, when we enlarge the pictures and plot the Hill statistics only for $k < 0.1\%$ of the sample size we can observe a much better fit. For the 1.8-stable sample Hill estimates are very close to the true value of α for $k \in (150, 400)$. But for the "hopeless" case of $\alpha = 1.95$ again there is no single value of k that gives the right value of the tail exponent. Yet this time the estimates are very close to the Levy-stable regime. Like for smaller samples, the Hill estimates agree quite well with log-log regression estimates of the true tail, see Figs. 3 and 5. For the extreme case of 10^8 observations the results are similar. The flat regions suggest almost the same values of α_{Hill} as did the Hill plots in Fig. 9. In fact the Hill plots for $k < 10\%$ of the sample size look so much alike, that in Fig. 10 we plotted only the counterparts of the right plots of Fig. 9, i.e. the enlarged plots for $k < 0.1\%$ of the sample size. We can see that for $\alpha = 1.8$ a good estimate is obtained only for $k = 25000, ..., 60000$ (i.e. for $k < 0.06\%$ of sample size), whereas for $\alpha = 1.95$ the estimate is in the Levy-stable regime only for $k < 20000$ (i.e. for $k < 0.02\%$ of sample size).

4 Conclusions

In the previous Section we showed that widely used tail index estimates (log-log linear regression and Hill) can give exponents well above the asymptotic limit for Levy-stable distributions with α close to 2. As a result, tail indices are significantly overestimated in samples of typical size. Only very large data sets (10^6 observations or more) exhibit the true tail behavior and decay as $x^{-\alpha}$. In practice, this means that in order to estimate α we must use high-frequency asset returns and analyze only the most outlying values. Otherwise, inference of the tail index may be strongly misleading and rejection of the Levy-stable regime unfounded.

Recently extremely large data sets have become available to the researchers. The largest ones studied in the literature so far (in the context of tail behavior) are (i) the 40 million data points record of 5 minute increments for 1000 U.S. companies during the two year period 1994-95 [13, 15] and (ii) the 1 million data points record of 1 minute increments for the Standard&Poor's 500 index during the 13-year period 1984-96 [13, 14]. However, the former data set is not homogeneous. It is formed out of 1000 data sets of 40000 records for individual companies. Therefore strong correlations, which can conceal the true nature of asset returns, are present in the data. The estimated tail indices for individual companies (see Fig. 1(b) in Ref. [15]) were found to range from $\alpha = 1.5$ to 5.5. As we have shown in the previous Section these values can be easily obtained for samples of Levy-stable distributed variables.

On the other hand, the second (ii) data set may be regarded as homogeneous [39]. The tails of the distribution of 1 minute returns were reported to decay in a power-law fashion with an exponent $\alpha = 2.95$ for positive and $\alpha = 2.75$ for negative observations [14]. In an earlier paper by the same authors [13] the tail exponents were reported to be 2.93 and 3.02, respectively, which shows that the log-log linear regression method is very sensitive to the choice of observations used. Moreover, in both papers the tails of the empirical cumulative distribution function curl upward for extreme returns [40], see Fig. 1(c) in [13] and Fig. 4(a) in [14]. This suggests that for very large and very small observations the distribution could be fitted by a power-law with a much smaller exponent. For example, if in Fig. 4(a) of [14] we plot an approximate regression line for negative returns in the region $20 \leq g \leq 100$ we find that the power-law exponent is less then 2. This may indicate that the tail exponents reported in both papers did not refer to the true tail behavior, but to the initial power-like decay (see Figs. 3 and 5) and that the rejection of the Levy-stable regime was not fully justified.

In this paper we have shown that the reported estimated tail exponent around 3 may very well indicate a Levy-stable distribution with $\alpha \approx 1.8$ [41]. This is consistent with earlier findings (for a review see [7]) where the returns of numerous financial assets (individual stocks, indices, FX rates, etc.) were reported to be Levy-stable distributed with $\alpha \in (1.65, 2]$. However, nothing we have said demonstrates that asset returns are indeed Levy-stably distributed. Although the analyzed tail index estimates are not sufficient to reject Levy-stability, by no means can we rule out a leptokurtic non-Levy-stable distribution that has power-law tails with $\alpha > 2$.

5   Acknowledgments

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[1] In the mathematical literature Levy-stable laws are called α-stable or just stable. Such a name has been assigned to these distributions because a sum of two independent random variables having a Levy-stable distribution with index α is again Levy-stable with the same index α. However, this invariance property does not hold for different α's, i.e. a sum of two independent Levy-stable random variables with different tail exponents is not Levy-stable.

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[41] When analyzing samples of typical size. For larger samples, e.g., of order 5×10^4, the tail exponent of 3 is obtained for $\alpha \approx 1.9$ rather than 1.8. For ever larger samples, the true tail behavior can be observed, see Fig. 3, 5 and 7.