The Differential Approach to Superlative Index Number Theory

by

William A. Barnett, Ki-Hong Choi, and Tara M. Sinclair

December 28, 2001

Forthcoming in the Hans Theil Memorial Special issue of the *Journal of Agricultural and Applied Economics*

William A. Barnett is Professor of Economics at Washington University. Ki-Hong Choi is at the National Pension Research Center in Seoul, Korea. Tara M. Sinclair is at Washington University.

Contact Author:

William A. Barnett
Economics Department
Washington University
Campus Box 1208
One Brookings Drive
Saint Louis, MO 63110
Phone: 314-935-4236
Fax: 314-935-4156
Email: barnett@wueconc.wustl.edu

Abstract

Diewert's (1976) "superlative" index numbers, defined to be exact for second order aggregator functions, unify index number theory with aggregation theory, but have been difficult to identify. We present a new approach to finding elements of this class. This new approach, related to that advocated by Henri Theil (1973), transforms candidate index numbers into growth rate form and explores convergence rates to the Divisia index. Since the Divisia index in continuous time is exact for any aggregator function, any discrete time index number that converges to the Divisia index and that has a third order remainder term is superlative.

Keywords: Divisia, index numbers, superlative indexes.

1. **Introduction**

According to Theil (1960, p. 464), "The subject of index numbers is one of the oldest in statistics and also, as regards the more specialized subject of cost of living index numbers, an old one in economics." Although an old subject, economists have long struggled to identify useful index numbers. The most influential selection criterion is that the index number be exact for an aggregator function that can produce a second order approximation to any twice continuously differentiable linearly homogeneous function. Diewert (1976) defined such index numbers to be "superlative." Superlative index numbers thus have known tracking ability relative to the exact aggregator functions of economic aggregation theory.

The class of superlative index numbers contains an infinite number of index numbers, since an infinite number of second order aggregator functions exist, but only a small number of index numbers in the superlative class have so far been found. The search process has previously involved finding an index number which is exact for a known second order algebraic aggregator function, or searching for a second order aggregator function for which a known index number is exact. No simple procedure has been found for either direction. For example, the minflex Laurent aggregator function, originated by Barnett and Lee (1985) over fifteen years ago, is known to be second order, but no one has succeeded in finding the index number that can track it exactly. In the other direction, Fisher (1911) proposed many index numbers in his famous book, but to the present day the aggregator functions tracked exactly by them remain unknown for most of those index numbers.

The Divisia continuous time index holds a prominent place in the literature because Ville, Hulten, Samuelson and Swamy, and Barnett and Serletis (p. 101-102) have shown that the Divisia line integral produces the unique exact index number formula for any neoclassical aggregator function. Similarly, the Divisia price index is the unique exact index number formula in continuous time for the neoclassical aggregator function's dual unit cost function. These results imply that the Divisia integral index is the prototype economic index number. For general use, however, the Divisia continuous time index must be adapted to apply to discrete data. A log-change form index is usual for all well known discrete time approximations to the Divisia index. This observation alone is not sufficient to determine the weight function. Thus there have been published large numbers of potential finite change approximations to the Divisia index. Each is in log change form, and they are differentiated by their weights.

We show that Theil's differential approach, which he used to support the Törnqvist index (1973), can be used systematically to determine which finite change approximations to the Divisia index are superlative. Since the Divisia line integral in continuous time is exact for any aggregator function, any superlative discrete time index number must: 1) converge to the Divisia index as the time intervals narrow and 2) have a third order remainder term for finite-change time intervals.. This is true regardless of whether or not we are capable of finding the second order aggregator function for which the discrete time index number is exact in discrete time.

To use this approach, it is necessary to be able to put candidate index numbers into growth rate form in discrete time (Theil's (1974) "Log-Change" form), so that convergence rates to the Divisia index can be explored. Using convergence theorems, which are widely available in mathematics, it becomes possible to identify large numbers or comparably good index numbers, and perhaps even to find new index numbers with better properties than the currently known index numbers.

In this paper, we will show that the most well known index numbers can be represented in log-change form. In the next section, we will consider the mean of order r class of index numbers. Then we will consider the quadratic mean of order r class of index numbers. In the conclusion we will discuss the interpretation of the weights in log-change form and suggest further research in this area.

2. **Log-Change Representation of the Mean of Order r Class of Index Numbers**

Let x_i^t be the quantity of good i during period t, and let p_i^t be its price. The mean of order r index of aggregate price change between periods 0 and 1, as defined by Allen and Diewert (1981), is characterized by its selection of the exponent r and superscript t. If we define the period t cost shares as $s_i^t \equiv \frac{p_i^t x_i^t}{p^t \cdot x^t}$ for i = 1, . . ., N, the *mean of order r index* of aggregate price change for r ≠ 0 between periods 0 and 1, using period t shares, is

$$\text{(1)} \qquad \mathrm{P}_{\mathrm{r,t}} \equiv \left[\sum_{i=1}^{n} s_i^t \left(\frac{p_i^1}{p_i^0} \right)^r \right]^{\frac{1}{r}}$$

where $\mathrm{P}_{\mathrm{r,t}} = \frac{\mathrm{P}_{\mathrm{r,t}}^1}{\mathrm{P}_{\mathrm{r,t}}^0}$ is the price change index, with $\mathrm{P}_{\mathrm{r,t}}^1$ being the price index level in period 1 and $\mathrm{P}_{\mathrm{r,t}}^0$ being the price index level in period 0.

Members of this class include the Laspeyres index (r = 1, t = 0) and the Paasche index (r = -1, t = 1). The mean of order r quantity index is defined analogously by interchanging the role of prices and quantities in the definition.

Theorem 1:

The mean of order r price index can be transformed into log-change form, and the sum of weights in log-change form is less than or equal to unity.

Proof:

If we take the natural logarithm of both sides of the mean of order r price index, we find:

$$\ln P_{r,t} = \frac{1}{r}\ln\sum_{i=1}^{n} s_i^t \left(\frac{p_i^1}{p_i^0}\right)^r = \frac{1}{r}\ln\frac{\sum_{i=1}^{n} v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r}{\sum_{k=1}^{n} v_k^t} \tag{2}$$

where $v_i^t \equiv p_i^t x_i^t$, and where $\ln \mathrm{P}_{\mathrm{r,t}} = \ln \mathrm{P}_{\mathrm{r,t}}^1 - \ln \mathrm{P}_{\mathrm{r,t}}^0$.

We now apply the concept of the log-mean, which was introduced by Vartia and Sato to the economic literature for x, y > 0:

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln\dfrac{x}{y}}, & \mathrm{x} \neq \mathrm{y} \\ \mathrm{x}, & \mathrm{x} = \mathrm{y} \end{cases} \tag{3}$$

For our equation we let x = $\sum_{i=1}^{n} v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r$ and we let y = $\sum_{k=1}^{n} v_k^t$. We then have

$$\frac{1}{r}\ln\frac{\sum_{i=1}^{n} v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r}{\sum_{k=1}^{n} v_k^t} = \frac{1}{r}\frac{\sum_{i=1}^{n} v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r - \sum_{i=1}^{n} v_i^t}{L\left[\sum_{k=1}^{n} v_k^t \left(\frac{p_k^1}{p_k^0}\right)^r, \sum_{k=1}^{n} v_k^t\right]} \tag{4}$$

$$= \frac{1}{r}\frac{\sum_{i=1}^{n}\left[v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r - v_i^t\right]}{L\left[\sum_{k=1}^{n} v_k^t \left(\frac{p_k^1}{p_k^0}\right)^r, \sum_{k=1}^{n} v_k^t\right]} = \frac{1}{r}\frac{\sum_{i=1}^{n} L\left[v_i^t \left(\frac{p_i^1}{p_i^0}\right)^r, v_i^t\right]\ln\left(\frac{p_i^1}{p_i^0}\right)^r}{L\left[\sum_{k=1}^{n} v_k^t \left(\frac{p_k^1}{p_k^0}\right)^r, \sum_{k=1}^{n} v_k^t\right]}$$

$$= \sum_{i=1}^{n} \frac{L\left[v_i^t\left(\frac{p_i^1}{p_i^0}\right)^r, v_i^t\right]}{L\left[\sum_{k=1}^{n} v_k^t\left(\frac{p_k^1}{p_k^0}\right)^r, \sum_{k=1}^{n} v_k^t\right]} \ln\left(\frac{p_i^1}{p_i^0}\right).$$

Letting

(5) $$w_i^{r,t} \equiv \frac{L\left[v_i^t\left(\frac{p_i^1}{p_i^0}\right)^r, v_i^t\right]}{L\left[\sum_{k=1}^{n} v_k^t\left(\frac{p_k^1}{p_k^0}\right)^r, \sum_{k=1}^{n} v_k^t\right]},$$

we have that

(6) $$l\text{n } \mathrm{P}_{\mathrm{r,t}} = \sum_{i=1}^{n} w_i^{r,t} \ln\left(\frac{\mathrm{p}_\mathrm{i}^1}{\mathrm{p}_\mathrm{i}^0}\right),$$

which is the log-change form of the mean of order r price index. The weight, $w_i^{r,t}$, can equivalently be written as:

(7) $$w_i^{r,t} \equiv s_i^t \frac{L\left[\left(\frac{p_i^1}{p_i^0}\right)^r, 1\right]}{L\left[\sum_{k=1}^{n} s_k^t\left(\frac{p_k^1}{p_k^0}\right)^r, 1\right]}.$$

From this form we can show that its sum is less than or equal to unity as follows:

(8) $$\sum_{i=1}^{n} w_i^{r,t} = \sum_{i=1}^{n} s_i^t \frac{L\left[\left(\frac{p_i^1}{p_i^0}\right)^r, 1\right]}{L\left[\sum_{k=1}^{n} s_k^t \left(\frac{p_k^1}{p_k^0}\right)^r, 1\right]} = \frac{\sum_{i=1}^{n} s_i^t L\left[\left(\frac{p_i^1}{p_i^0}\right)^r, 1\right]}{L\left[\sum_{k=1}^{n} s_k^t \left(\frac{p_k^1}{p_k^0}\right)^r, 1\right]}$$

But $L(x,1) = \frac{x-1}{\ln x}$ is concave, as proven in the Appendix. Hence by Jensen's inequality, $\sum_{i=1}^{n} w_i^{r,t} \leq 1$, with equality only if r = 0 or $\frac{p_i^1}{p_i^0} = 1$ for all i. Since the log-change approximation to the Divisia index is based on the Weighted Mean Value Theorem (Fulks(1978), p. 162), we have the requirement that the discrete time interval should be as narrow as data allow. As $\frac{p_i^1}{p_i^0} \to 1$ for all i, the sum of the weights approaches unity. *Q.E.D.*

Note that this encourages the use of the chain method introduced by Alfred Marshall and described by Frisch. Diewert (1978) also argues for the use of this method.

Examples:

We can write the log-change form of the Laspeyres index of aggregate price change between periods 0 and 1 (r = 1, t = 0) as follows:

(9) $$\ln \mathrm{P}_{\text{Laspeyres}} = \sum_{i=1}^{n} \frac{L\left[v_i^0 \left(\frac{p_i^1}{p_i^0}\right), v_i^0\right]}{L\left[\sum_{k=1}^{n} v_k^0 \left(\frac{p_k^1}{p_k^0}\right), \sum_{k=1}^{n} v_k^0\right]} \ln\left(\frac{p_i^1}{p_i^0}\right),$$

where $\ln \mathrm{P}_{\text{Laspeyres}} = \ln \mathrm{P}^1_{\text{Laspeyres}} - \ln \mathrm{P}^0_{\text{Laspeyres}}$, and where the Laspeyres weights are

$$w_i^{Laspeyres} = \frac{L\left[v_i^0\left(\frac{p_i^1}{p_i^0}\right), v_i^0\right]}{L\left[\sum_{k=1}^{n} v_k^0\left(\frac{p_k^1}{p_k^0}\right), \sum_{k=1}^{n} v_k^0\right]}.$$

Similarly, we can write the log-change form of the Paasche price index (r = -1, t = 1) as follows:

(10) $$\ln \mathrm{P}_{\mathrm{Paasche}} = \sum_{i=1}^{n} \frac{L\left[v_i^1\left(\frac{p_i^0}{p_i^1}\right), v_i^1\right]}{L\left[\sum_{k=1}^{n} v_k^1\left(\frac{p_k^0}{p_k^1}\right), \sum_{k=1}^{n} v_k^1\right]} \ln\left(\frac{p_i^1}{p_i^0}\right),$$

where $\ln \mathrm{P}_{\mathrm{Paasche}} = \ln \mathrm{P}_{\mathrm{Paasche}}^1 - \ln \mathrm{P}_{\mathrm{Paasche}}^0$, and where the Paasche weights are

$$w_i^{Paasche} = \frac{L\left[v_i^1\left(\frac{p_i^0}{p_i^1}\right), v_i^1\right]}{L\left[\sum_{k=1}^{n} v_k^1\left(\frac{p_k^0}{p_k^1}\right), \sum_{k=1}^{n} v_k^1\right]}.$$

3. **Log-Change Representation of the Quadratic Mean of Order r Class of Index Numbers**

The quadratic mean of order r index is closely related to the mean of order r index from above. Diewert (1976) defines the *quadratic mean of order r price change index* in terms of the mean of order r price change index as follows:

(11) $$P_r \equiv \left(P_{\frac{r}{2},0} P_{-\frac{r}{2},1}\right)^{\frac{1}{2}} \equiv \left(\left[\sum_{i=1}^{n} s_i^0\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}}\right]^{\frac{2}{r}}\left[\sum_{i=1}^{n} s_i^1\left(\frac{p_i^1}{p_i^0}\right)^{-\frac{r}{2}}\right]^{-\frac{2}{r}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{\left[\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0} \right)^{\frac{r}{2}} \right]}{\left[\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^1}{p_i^0} \right)^{-\frac{r}{2}} \right]} \right)^{\frac{1}{r}},$$

for $r \neq 0$, where $P_r = \frac{P_r^1}{P_r^0}$ is the price change index, with P_r^1 being the price index level in period 1 and P_r^0 being the price index level in period 0.

Diewert (1976) has shown that the quadratic mean of order r class of indexes is superlative for all r. That class is the most general superlative index number specification known and includes the Fisher ideal index and the Törnqvist index.

Corollary to Theorem 1:

The quadratic mean of order price index can be transformed into log-change form, and the sum of weights in log-change form is less than or equal to unity.

Proof:

The proof is a simple algebraic manipulation applying the relationship between the quadratic mean of order r price index, the mean of order r price index, and our result from Theorem 1. Note that

(12) $$\ln P_r = \frac{1}{2}\ln\left(P_{\frac{r}{2},0}\right) + \frac{1}{2}\ln\left(P_{-\frac{r}{2},1}\right) = \frac{1}{2}\left[\sum_{i=1}^{n} w_i^{\frac{r}{2},0} \ln\left(\frac{p_i^1}{p_i^0}\right) + \sum_{i=1}^{n} w_i^{-\frac{r}{2},1} \ln\left(\frac{p_i^1}{p_i^0}\right)\right]$$

so that

(13) $$\ln P_r = \sum_{i=1}^{n} \frac{1}{2}\left(w_i^{\frac{r}{2},0} + w_i^{-\frac{r}{2},1} \right) \ln\left(\frac{\mathrm{p}_\mathrm{i}^1}{\mathrm{p}_\mathrm{i}^0} \right) = \sum_{i=1}^{n} w_i^r \ln\left(\frac{\mathrm{p}_\mathrm{i}^1}{\mathrm{p}_\mathrm{i}^0} \right),$$

where

(14) $$w_i^r = \frac{1}{2}\left(w_i^{\frac{r}{2},0} + w_i^{-\frac{r}{2},1} \right).$$

We can expand w_i^r to determine the value of its sum:

(15) $$w_i^r \equiv \frac{1}{2}\left(s_i^0 \frac{L\left[\left(\frac{p_i^1}{p_i^0} \right)^{\frac{r}{2}}, 1 \right]}{L\left[\sum_{k=1}^{n} s_k^0 \left(\frac{p_k^1}{p_k^0} \right)^{\frac{r}{2}}, 1 \right]} + s_i^1 \frac{L\left[\left(\frac{p_i^1}{p_i^0} \right)^{-\frac{r}{2}}, 1 \right]}{L\left[\sum_{k=1}^{n} s_k^1 \left(\frac{p_k^1}{p_k^0} \right)^{-\frac{r}{2}}, 1 \right]} \right).$$

If we take the summation of both sides we obtain:

(16) $$\sum_{i=1}^{n} w_i^r \equiv \frac{1}{2}\left(\frac{\sum_{i=1}^{n} s_i^0 L\left[\left(\frac{p_i^1}{p_i^0} \right)^{\frac{r}{2}}, 1 \right]}{L\left[\sum_{k=1}^{n} s_k^0 \left(\frac{p_k^1}{p_k^0} \right)^{\frac{r}{2}}, 1 \right]} + \frac{\sum_{i=1}^{n} s_i^1 L\left[\left(\frac{p_i^1}{p_i^0} \right)^{-\frac{r}{2}}, 1 \right]}{L\left[\sum_{k=1}^{n} s_k^1 \left(\frac{p_k^1}{p_k^0} \right)^{-\frac{r}{2}}, 1 \right]} \right).$$

As in the case of the mean of order r index, each of the two terms within the outer parentheses on the right hand side is less than or equal to one. Hence the sum of those terms, divided by two, is also less than or equal to one and approaches one as $\frac{p_i^1}{p_i^0} \to 1$ for all i. *Q.E.D.*

Examples:

We can write the log-change form of the Fisher ideal price index (r = 2) as follows:

(17) $$\ln \mathrm{P}_{\mathrm{Fisher}} = \sum_{i=1}^{n} \frac{1}{2}\left(w_i^{Laspeyres} + w_i^{Paasche}\right)\ln\left(\frac{p_i^1}{p_i^0}\right),$$

where $\ln \mathrm{P}_{\mathrm{Fisher}} = \ln \mathrm{P}^1_{\mathrm{Fisher}} - \ln \mathrm{P}^0_{\mathrm{Fisher}}$. We also know that the Törnqvist index is a limiting member of this class (for r→0). Since we did not consider the case of r→0 above, we now show that this index also can be represented in log-change form. For r = 0, Allen and Diewert (1981) defined the quadratic mean of order 0 price change index as:

(18) $$P_0 \equiv \prod_{i=1}^{n}\left(\frac{p_i^1}{p_i^0}\right)^{\left(\frac{1}{2}s_i^0 + \frac{1}{2}s_i^1\right)}$$

where $P_0 = \frac{P_0^1}{P_0^0}$.

Result:

P_0 can be represented in log-change form, and the weights then sum to 1.

Proof:

(19) $$\ln P_0 = \ln \prod_{i=1}^{n}\left(\frac{p_i^1}{p_i^0}\right)^{\left(\frac{1}{2}s_i^0 + \frac{1}{2}s_i^1\right)} = \sum_{i=1}^{n}\frac{1}{2}\left(s_i^0 + s_i^1\right)\ln\left(\frac{p_i^1}{p_i^0}\right) = \sum_{i=1}^{n} \mathrm{w}_{\mathrm{i}}^0 \ln\left(\frac{p_i^1}{p_i^0}\right),$$

where

(20) $$w_i^0 \equiv \frac{1}{2}\left(s_i^0 + s_i^1\right)$$

We can see from the definition of the weights, $\mathrm{w}_{\mathrm{i}}^0$ (i = 1, . . ., N), that their sum in this special case becomes exactly unity. *Q.E.D.*

4. **Conclusion**

We have shown that two large classes of economic index numbers can be represented in log-change form. This provides a new method to determine whether index numbers are superlative. Instead of searching for the second order aggregator function for which the discrete time index number is exact in discrete time, we can test the suggested index number's convergence to the Divisia index in log-change form. If an index converges to the Divisia index as the time intervals narrow and has a third order remainder term for finite-change time intervals, then the index is superlative.

The log-change form provides a unified view of index number formulas and their convergence properties. There also is a useful interpretation of index numbers in this form. Recall the form:

$$\ln \mathrm{P} = \sum_{\mathrm{i}=1}^{\mathrm{n}} w_i \ln\left(\frac{\mathrm{p}_\mathrm{i}^1}{\mathrm{p}_\mathrm{i}^0}\right), \tag{21}$$

where $\ln \mathrm{P} = \ln \mathrm{P}^1 - \ln \mathrm{P}^0$. Equation (21) is an additive decomposition (Törnqvist, Vartia, and Vartia (1985)) of the global rate of growth, ln P, into each contributing factor $w_i \ln\left(\frac{\mathrm{p}_\mathrm{i}^1}{\mathrm{p}_\mathrm{i}^0}\right)$.

These results suggest the potential productivity of further research investigating new index numbers by this method to see if they are superlative, or better than superlative with remainder terms of order higher than 3.

Appendix

Let f(x) = L(x,1), where $L(x,1) = \frac{x-1}{\ln x}$ is defined as in the proof of Theorem 1. We now prove the following lemma.

Lemma 1: f(x) is concave for all $x \neq 1$, x>0.

Proof: The second derivative of f is

$$f''(x) = \frac{-x\ln x - \ln x + 2x - 2}{x^2(\ln x)^3}.$$

Letting $d(x) = x^2(\ln x)^3$ and $h(x) = -x\ln x - \ln x + 2x - 2$, we obtain

$$f''(x) = \frac{h(x)}{d(x)}.$$

Clearly $d(x)$ is negative for $0 < x < 1$ and positive for $x > 1$.

Observe that $h(1) = 0$, $h'(1) = 0$, and $h''(1) = 0$, since $h'(x) = -\ln x - 1/x + 1$ and $h''(x) = (1-x)/x^2$. Thus we know that $x = 1$ is an inflection point (critical point) of the curve $h(x)$. But $h''(x) < 0$ for x>1, and $h''(x) > 0$ for 0<x<1, so that h(x) is strictly concave for x>1 and strictly convex for 0<x<1. Hence h(x) is negative for x>1 and positive for $0 < x < 1$.

It follows immediately that $f''(x) < 0$ for all positive $x \neq 1$. *Q.E.D.*

This result can be generalized to the full logarithmic mean function $L(x,y) = \dfrac{x-y}{\ln x - \ln y}$.

We are indebted to W. Erwin Diewert for the following proof provided to us through private correspondence.

Corollary to Lemma 1: L(x,y) is concave for all x,y>0, $x \neq y$.

Proof: Consider the Hessian matrix, H(x,y), of second order partial derivatives of L(x,y). By an argument analogous to that used in the proof of Lemma 1, it follows that the diagonal elements of H(x,y) are strictly negative for $x \neq y$. But since L(x,y) is linearly homogeneous, the determinant of H(x,y) is zero. See Hadar (1971, eq. 5.32, p. 71). At all points off of the ray $x \neq y$, the necessary and sufficient conditions are therefore satisfied for H(x,y) to be negative semidefinite. *Q.E.D.*

References

Allen, R. and W.E. Diewert. "Direct Versus Implicit Superlative Index Number Formulae." *Review of Economics and Statistics* 63(August, 1981):430-435.

Barnett, W.A. and A. Serletis. *The Theory of Monetary Aggregation*. Amsterdam: North Holland, 2000.

Barnett, W.A. and Y. Lee. "The Global Properties of the Minflex Laurent, Generalized Leontief, and Translog Flexible Functional Forms." *Econometrica* 53(November 1985):1421-1438.

Diewert, W.E. "Exact and Superlative Index Numbers." *Journal of Econometrics* 4(May 1976):115-145.

Diewert, W.E. "Superlative Index Numbers and Consistency In Aggregation." *Econometrica* 46(July 1978):883-900.

Fisher, I. *The Purchasing Power of Money*. London: Macmillan, 1911.

Frisch, R. "Annual Survey of General Economic Theory: The Problem of Index Numbers." *Econometrica* 4(January 1936):1-38.

Fulks, W. *Advanced Calculus*. New York: John Wiley & Sons, 1978.

Hadar, J. *Mathematical Theory of Economic Behavior*. Reading, MA: Addison-Wesley, 1971.

Hulten, C. "Divisia Index Numbers." *Econometrica* 41(November 1973):1017-1026.

Samuelson, P.A. and S. Swamy. "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis." *American Economic Review* 64(September 1974):566-593.

Sato, K. "The Ideal Log-Change Index Number." *The Review of Economics and Statistics* 58(May 1976):223-228.

Theil, H. "A New Index Number Formula." *The Review of Economics and Statistics* 55(November 1973):498-502.

Theil, H. "Best Linear Index Numbers of Prices and Quantities." *Econometrica* 28(April 1960):464-480.

Theil, H. "More on Log-Change Index Numbers." *The Review of Economics and Statistics* 56(November 1974):552-554.

Törnqvist, L., P. Vartia, and Y.O. Vartia. "How Should Relative Change Be Measured?" *The American Statistician* 39(1985):43-46.

Vartia, Y.O. "Ideal Log-Change Index Numbers." *Scandinavian Journal of Statistics* 3(September 1976):121-126.

Newman, P.K. and J. Ville. "The Existence Conditions of A Total Utility Function." *The Review of Economic Studies* 19(1951-1952):123-128.