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U-shaped Paths of Consumption and Physical Capital in Lucas-type Growth Models

by

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Date August 2002

Abstract This paper considers transitional dynamics of consumption and physical capital in Lucas-type growth models. We find that when the ratio of physical to human capital is sufficiently high, it is optimal for both consumption and physical capital to fall for a finite period and then rise along their transition paths. Endogenous growth models may therefore not be able to generate a sustainable path of development when the steady state coexists with the transitional dynamics. According to the extent of consumption smoothing, we characterize the stages of transition in three different growth regimes. Particularly in the normal and paradoxical cases, dynamics of the Lucas model is contrasted with that of Ramsey.

Keywords Lucas growth model, transitional dynamics, physical to human capital ratio, sustainability

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U-shaped Paths of Consumption and Physical Capital in the Lucas-type Growth Models

Farhad Nili

1 Introduction

Multi-sector endogenous growth models have received great attention in recent years, though, due to intractability, the dynamics of these models are not yet well understood. Because they initially proved to be intractable, "much of the analysis was restricted to balanced-growth paths ..., or to analyzing transitional dynamics using numerical simulation methods." Turnovsky(2000, p.502)

It is a common practice to abstract from transitional dynamics, by assuming that the steady state growth path can be reached instantaneously; Barro and Sala-i-Martin(1995, ch. 6 and 7) and Aghion and Howitt(1998, ch.5). This, from an analytical standpoint, provides a great deal of convenience. It makes the analysis so tractable and in some situations approximates the reality quite well. Moreover, by taking advantage of this simplification, "problems such as time consistency of the optimal policy do not arise." Turnovsky(p.487)

The convenience, however, comes at some costs. The steady state analysis describes a situation which is never attainable in finite time. Quantitative analysis shows that the actual economies move approximately 4% of their remaining distance toward their steady state each year; Romer(1996, sec. 1.5). They spend most of their time, therefore, on the transition rather than steady state path. The argument that all the variables in the economy grow at their long-run growth rates, may therefore be misleading in some situations. Furthermore, the steady state analysis does not allow for the model to accommodate policy changes and other shocks that are continually occurring. It is therefore important to extend the growth models, where necessary, to allow for the coexistence of both steady state and transitional paths.

Most of the studies that do not restrict their analysis to steady state, either tackle the transitional dynamics by numerical methods; Mulligan and Sala-i-Martin(1993), or approximate the original dynamics by the behaviour of the linearized system around the steady state; Bond et al.(1996). The current study, instead, is not restricted to the vicinity of the steady state. Our treatment is also based merely on analytical methods.

We examine the transitional dynamics of a two-sector endogenous growth model in the tradition of Lucas(1988). We focus on the dynamics of consumption and physical capital. Although investigating the transitional dynamics along the whole off-balanced path is not in general tractable, we show that the Lucas assumption, by which only one type of capital is allocated across both sectors, makes the analysis tractable.

According to our findings, in a Lucas economy, with a high enough ratio of physical to human capital, both consumption and physical capital fall along their off-balanced paths. They decline for a finite period and then rise on their transition, toward the steady state. When the output-capital ratio is far short of its steady state and consumption smoothing is strong, four episodes occur during transition. First, both consumption and physical capital fall. Next consumption rises but physical capital still decumulates. In the third phase, both variables rise with different rates and finally, at the steady state they grow at a common and constant rate. The time profile of consumption and physical capital is therefore U-shaped, where the minimum of physical capital lags behind that of consumption. A numerical exercise suggests that the *falling period* is not very far from the vicinity of the steady state.

The second stage, in which consumption and physical capital move in opposite directions, distinguishes the Lucas-type models from that of Ramsey. In the *exogenous growth regime* where the Lucas model reproduces the dynamics of the Ramsey, this stage disappears and consumption and physical capital move in a parallel fashion along their U-shaped paths. In the *paradoxical regime*, in which agents are very impatient, however, in the second stage, the fall of consumption coincides with a rise in physical capital while the other stages are repeated in the same manner.

In the R&D-based endogenous growth models, accumulation of knowledge through innovation, plays the same role as the human capital development in the Lucas setting. Regardless of their different microfoundations, the two groups of human capital and R&D-based models, share the same reduced form in the context of optimal growth theory, so long as they are subject to the Lucas assumption by which one sector specializes in only one type of capital. This implies that our results are applicable to the R&D-based models too. This includes the "expanding variety" model of Romer and the "creative destruction" model of Aghion and Howitt.

An application of the U-shaped path of consumption and physical capital, is to contrast two notions of optimality and sustainability in Lucas-type growth models in a concrete way. Optimality, according to this argument, may depart from sustainability along the transition path though growth is sustained at the steady state.

Mulligan and Sala-i-Martin(1993) attempt to uncover the transitional dynamics of the two-sector endogenous growth models. They consider, in their numerical investigation, the possibility of falling of consumption and physical capital during transition path in the Lucas model. They do not characterize however the problem. We diagnose here through the falling symptom completely by means of analytical methods.

Caballé and Santos(1993) also address the decline of both consumption and physical capital in an economy endowed, in relative terms, with a great amount of physical capital. They conclude then, physical capital and consumption in the two-sector endogenous growth model respond to the increment in physical capital in a similar qualitative way as in the Ramsey model because "economies with high ratios of physical to human capital will always decumulate physical capital, and economies with low ratios of physical to human capital will always increase their holdings of physical capital." (p.1064) Our findings however illustrates that decline of consumption and physical capital in the Lucas model is different from that of Ramsey where capital exceeds the golden rule.

We present the benchmark model of a Lucas economy in next section. This is followed by exploring the dynamics of consumption and physical capital in section 3. According to the extent of consumption smoothing, we examine our findings in two different regimes in section 4. Section 5 extends the model to the case where there is an externality from human capital accumulation. In the light of our findings, we then reexamine the concept of sustainability in section 6. We conclude in section 7.

2 The model

2.1 Specification

Consider a two-sector endogenous growth model in the tradition of Lucas where labour in efficiency units, is reproduced in an unbounded fashion through education. Preferences are isoelastic and technology of output product is Cobb-Douglas. Population is taken as constant. There exists an external effect from the social stock of human capital to the productivity of individual agents and, according to a crucial assumption, education depends only on the human capital and the allocation of time between working and schooling. In other words, human capital is the only asset to be allocated across sectors. Lucas only considers the balanced growth path, along which all level variables grow at a constant rate.

As a benchmark model to begin with, let us first abstract from externalities from human capital accumulation. Because the decentralized version of the model, in this case, may be replicated through the planner's optimal solution, we shall limit without loss of generality our analysis to the planner's problem. In this regard our model is close to those of Barro and Sala-i-Martin(1995, sec. 5.2.2) and Arnold(2000).

Suppose the size of population is normalized to one. The objective of the planner is to maximize intertemporal isoelastic utility of the representative agent

$$W = \int_0^\infty \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \,, \tag{1}$$

where $C(t)$ denotes consumption, $\rho > 0$ is the discount rate, and $1/\sigma > 0$ is the intertemporal elasticity of consumption.

Output is determined by the Cobb-Douglass technology

$$Y = AK^\alpha (uH)^{1-\alpha}, \tag{2}$$

where K is physical capital, H denotes human capital, $u \in [0, 1]$ is the portion of time devoted to production of the final output, and $\alpha \in (0, 1)$ is the elasticity of output with respect to the physical capital. The final good is consumed, invested to accumulate physical capital or to replace depreciated old capital. Hence

$$\dot{K} = Y - C - \delta K. \tag{3}$$

Although in some literature e.g. Bond et al.(1996, p.153) and Turnovsky(2000, p.230), disinvestment is feasible, in line with mainstream, we suppose that investment is irreversible and hence gross investment is nonnegative[1]. This requires $\dot{K} + \delta K \geq 0$.

The representative producer-consumer supplies one unit of labour inelastically and lives forever. She devotes $1-u$ fraction of her effort to the development of human capital, referred to as education. Then

$$\dot{H} = B(1-u)H - \delta H, \tag{4}$$

[1] See Barro and Sala-i-Martin(1995, ch.5) for an extensive discussion on the issue.

where $B > 0$ is the productivity of education. This indicates that development of human capital only depends on the proportion of time spent on education and the pre-accumulated human capital. In particular it does not require the units of final output and hence physical capital. We also assume that both types of capital depreciate at a common rate[2].

2.2 Optimal growth and the steady state

The central planner chooses level of C and u such that W is maximized subject to the law of motion of K and H and given values for $K(0)$ and $H(0)$. Two decisions are made here: the allocation of output between consumption and investment in physical capital, and the allocation of time between working and education.

An optimal solution for this economy is a path, $\{C(t), K(t), H(t), u(t)\}$ which maximizes (1) subject to the constraints (3) and (4) and transversality conditions corresponding to K and H. We firstly concentrate on the steady state where the level variables grow at a common rate.

Definition 1 *The steady state is a path, along which, C, Y, K and H grow at constant rates and u remains unchanged.*

Lucas shows that in the absence of population growth and externality, all level variables share a common steady state rate of growth equal to

$$\widetilde{g} = B(1 - \widetilde{u}) - \delta, \tag{5}$$

where a tilde over a variable refers to its steady state.

It is more convenient therefore to transform the variables into a set of new variables $z \equiv Y/K$ and $\chi \equiv C/K$ that are stationary at the steady state. These, with u, form *fundamentals* of the model. A steady state, or balanced growth path, can then be defined as a situation where $\dot{z} = \dot{\chi} = \dot{u} = 0$, so that output, consumption and the two types of capital grow at a common rate while the work effort will be constant.

The problem of optimal growth of the Lucas model, has been solved by Barro and Sala-i-Martin(1995, sec.5.2), Ortigueira(1998) and Arnold(2000). The dynamics of the economy can be expressed by a set of linear growth differential equations (see appendix B for details) where for any variable like y, $g_y \equiv \dot{y}/y$ denotes its exponential rate of growth and $\lambda = (\alpha - 1)B/\alpha$,

$$\begin{bmatrix} g_z \\ g_\chi \\ g_u \end{bmatrix} = \begin{bmatrix} \alpha - 1 & 0 & 0 \\ \alpha/\sigma - 1 & 1 & 0 \\ 0 & -1 & B \end{bmatrix} \begin{bmatrix} z \\ \chi \\ u \end{bmatrix} - \begin{bmatrix} \lambda \\ (\rho + \delta)/\sigma - \delta \\ \lambda \end{bmatrix}. \tag{6}$$

In a compact form we have

$$[g_z, g_\chi, g_u]^T = Mx - [\lambda, (\rho + \delta)/\sigma - \delta, \lambda]^T,$$

where $x = [z, \chi, u]^T$ is the vector of fundamentals. Since the matrix of coefficient, M, has a nonzero determinant, equal to $(\alpha - 1)B$, it is nonsingular. As a

[2] The depreciation of human capital on the individual level reflects depreciation caused by forgetting. It also reflects the imperfections in the intergenerational transmission of skills.

result there exists a unique steady state defined by[3]

$$\widetilde{x} = M^{-1}\left[\lambda, (\rho+\delta)/\sigma - \delta, \lambda\right]^T .$$

Regarding the properties of the system (6), one should note that the model has a block recursive structure. The dynamics of the output-capital ratio, z is independent of χ and u. Further, the dynamics of the consumption-capital ratio, χ is independent of u. This substantially simplifies the analysis of the transitional dynamics.

Solving for the steady state value of the fundamental variables, one obtains

$$\begin{aligned} \widetilde{z} &= B/\alpha, \\ \widetilde{\chi} &= B/\alpha - (B-\rho-\delta)/\sigma - \delta, \\ \widetilde{u} &= 1 - (B-\rho-\delta)/B\sigma - \delta/B. \end{aligned} \tag{7}$$

From this, in conjunction with (6), one can reproduce Eqs. (5.31) - (5.33) of Barro and Sala-i-Martin(1995) as

$$\left[g_z, g_\chi, g_u\right]^T = M \cdot \left[z-\widetilde{z}, \chi-\widetilde{\chi}, u-\widetilde{u}\right]^T . \tag{8}$$

Plugging the steady state value of work effort, $\widetilde{u}$ into (5) gives the balanced rate of growth of the economy as

$$\widetilde{g} = (B-\rho-\delta)/\sigma. \tag{9}$$

Sustainability of the long run growth requires $\rho+\delta < B$ and the transversality condition corresponding to K implies $\widetilde{g} < B-\delta$. Two requirements are combined in the following condition.

Condition 1 *For the balanced rate of growth to sustain and also to meet the transversality conditions, one requires*

$$0 < B-\rho-\delta < \sigma(B-\delta). \tag{10}$$

This is also the sufficient condition for $\widetilde{\chi} > 0$, and necessary and sufficient condition for $\widetilde{u} \in (0,1)$, i.e. for the steady state to be well defined.

The effect of changes in parameters of the model on the steady state values $(\widetilde{z}, \widetilde{\chi}, \widetilde{u})$ and $\widetilde{g}$ is reported in Mulligan and Sala-i-Martin(1993, table 1).

2.3 Local dynamics around the steady state

Local dynamics of the system (6) in the vicinity of the steady state (7), can be found by linearizing the system of differential equations around $\widetilde{x}$. This can be summarized as

$$\dot{x} \simeq \widetilde{M} \cdot (x-\widetilde{x}), \tag{11}$$

where $\widetilde{M} = [\widetilde{m}_{ij}]$ is a 3×3 Jacobian matrix for which we have $\widetilde{m}_{ij} = m_{ij}\cdot\widetilde{x}_i$ and m_{ij} is the entry on *i-th* row and *j-th* column of M.

[3]See Bond et al.(1996) on the existence and uniqueness in a general setting.

By definition $\widetilde{M}$ is lower triangular like M and the eigenvalues, defined by $\lambda(.)$, coincide with its diagonal entries as

$$\lambda(\widetilde{M}) = \{ \ (\alpha-1)B/\alpha, \ B/\alpha - (B-\rho)/\sigma, \ B - (B-\rho)/\sigma \ \}.$$

Given (10) and the range of parameters of the model, there is only one negative eigenvalue, $\lambda = (\alpha-1)\,B/\alpha$, whose magnitude determines the speed of convergence in the vicinity of the steady state. This shows that (11) describes a one dimensional stable saddle path.

Starting from a given initial value of the average productivity of physical capital $z(0) = z_0$, the stable dynamic adjustment path is described around the steady state by

$$x_t - \widetilde{x} = (z_0 - \widetilde{z})(z_t/z_0) \cdot V \exp(\lambda t),$$

where $V = (1, v_2, v_3)^T$ is the eigenvector corresponding to the stable eigenvalue λ, that rules out the unstable paths.

The first equation in (6) presents a self contained differential equation for z which gives

$$z_t = \widetilde{z}.z_0 \left[z_0 - (z_0 - \widetilde{z}) \exp(\lambda t)\right]^{-1}.$$

This, for $z_0 < \widetilde{z}$, is logistic while for $z_0 > \widetilde{z}$, it declines exponentially toward $\widetilde{z}$. Moreover, since there is no restriction on the convergence of z to $\widetilde{z}$ and it attracts any $z > 0$, then stability is global rather than local; Arnold(2000).

We summarize our discussions about the optimal model described by Eqs. (1)-(4) in the following statements:

- There exists a unique steady state that is completely described by known parameters.
- The steady state is globally saddle stable.
- The speed of convergence, $-\lambda = (1-\alpha)\,B/\alpha$, depends solely on technical parameters.

2.4 Policy functions and growth regimes

The one-dimensional saddle stable arm can be denoted by the curve

$$(z, \chi(z), u(z)) : R_+ \longrightarrow R_+^2 \times [0,1],$$

that passes through the steady state, i.e. $\chi(\widetilde{z}) = \widetilde{\chi}$, and $u(\widetilde{z}) = \widetilde{u}$. The slope of the curve at the steady state also distinguishes the stable path from unstable ones, i.e. $\chi'(\widetilde{z}) = v_2$, and $u'(\widetilde{z}) = v_3$, where

$$v_2 = \left(\frac{1-\alpha/\sigma}{\widetilde{\chi}-\lambda}\right)\widetilde{\chi}, \qquad \text{and} \qquad v_3 = \left(\frac{1-\alpha/\sigma}{\widetilde{\chi}-\lambda}\right)\widetilde{u}. \tag{12}$$

Since according to Eq. (6), the dynamics of consumption and physical capital only depends on output-capital and consumption-capital ratios, from now on we only focus on the projection of the saddle path on the (z, χ) plane.

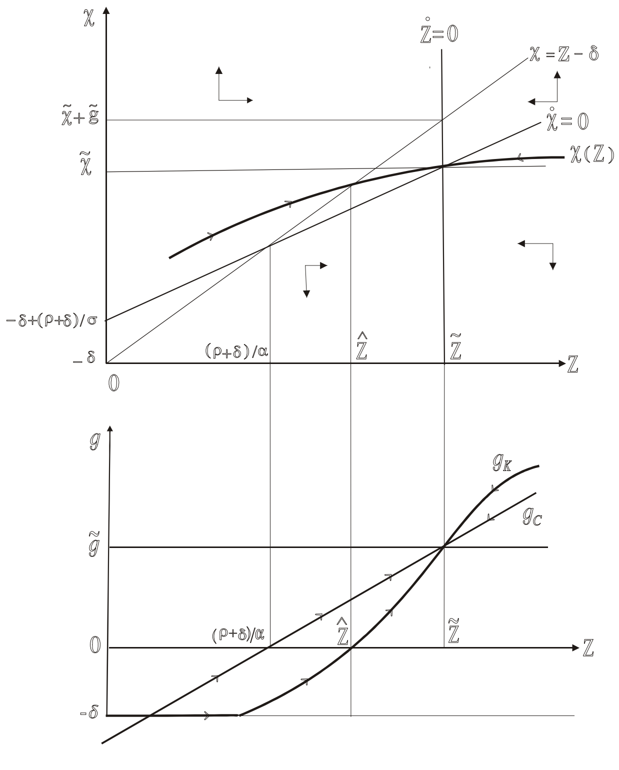

Figure 1: Phase diagrame of z and χ (upper panel) and g_C and g_K (lower panel)

Regarding the relative size of α and σ, three cases are distinguished in the literature; Caballé and Santos(1993). For the immediate purpose we consider the case where $\alpha < \sigma$ which is more interesting in practice. The cases where $\sigma \leq \alpha$ are discussed in section 5.

If $\alpha < \sigma$, the locus $\dot{\chi} = 0$ is upward sloping with slope $1 - \alpha/\sigma > 0$. For $0 \leq z_0 < \widetilde{z}$ we then have

$$(\rho + \delta)/\sigma - \delta + (1 - \alpha/\sigma)z < \chi(z) < \widetilde{\chi}\,.$$

Since the locus $\dot{z} = 0$ is vertical and stable, for $z_0 < \widetilde{z}$ (respectively $z_0 > \widetilde{z}$) , $z(t)$ converges monotonically to $\widetilde{z}$ from left (respective from right). The phase diagram in this case is depicted in the upper panel of figure 1.[4]

Mulligan and Sala-i-Martin(1993) identify three forces that govern the dynamics of the model off the steady state. Without loss of generality consider the case where $z_0 < \widetilde{z}$. This means that the economy is endowed with a relatively scarce level of human capital and the productivity of physical capital is low.

Firstly, owing to the global stability of the steady state, there exists a convergence mechanism that forces the economy from imbalances between physical and human capital to the balanced path. This is described by the dynamics of z alone.

In addition, there are two other forces. Convergence may be supported through consuming more and investing less in physical capital or via faster accumulation of human capital by devoting more time to schooling and working less. The stronger the consumption-smoothing effect, people dislike the former option. On the other hand, when H/K is low, the wage rate in the output sector is high, discouraging agents from schooling by lowering u.

When the technology of producing the final output is Cobb-Douglas, like the model at hand, the relative size of the intertemporal elasticity of substitution in consumption and the share of physical capital, account for the growth regime. The smaller σ, the stronger is the second effect. On the other hand, the larger α, the more powerful is the third force. The relative position of these two parameters determines how these three forces interact and which effect dominates.

If $\alpha < \sigma$, the consumption-smoothing effect dominates. Low z corresponds with low χ and u, meaning that the policy functions are upward sloping. In this case, the transition from relatively high levels of physical capital is accomplished through more schooling rather than higher saving.

When $\alpha = \sigma$, the two forces cancel out and only the convergence effect takes place. In this case the policy function is horizontal meaning that a constant proportion of time is devoted to education and the human capital grows at an exogenous rate. The Lucas model, as a result, replicates the dynamics of the Ramsey model where education takes the role of exogenous technical progress.

Finally if $\alpha > \sigma$, low values of z correspond with high χ and u, and policy functions are downward sloping. In this case, besides the convergence effect, transition to the steady state is governed mostly by changes in consumption patterns rather than by working effort. In particular, the dynamics of an economy starting from $z_0 < \widetilde{z}$ to the steady state in this regime is accomplished through less saving rather than supplying more effort.

[4] In comparison with figure 5.4 of Barro and Sala-i-Martin(1995), this is augmented with the required details to enable us extracting the transition path of rate of growth of consumption and physical capital along the whole possible range of output-capital ratio.

These three regimes are called the *normal*, *exogenous* and *paradoxical growth* cases respectively; Caballé and Santos(1993) and Ladron-de-Guerara et. al. (1997). Although, according to the empirical evidence, the Cobb-Douglas technology is in favour of the normal case, in a more general setting the numerical exercise of Caballé and Santos do not rule out other possibilities. Condition (10) on the other hand, imposes a lower bound on σ equal to $1-\rho/(B-\delta)$. This further limits the possibility of other cases in the Cobb-Douglas framework. For the rest of this study we follow the normal case until section 5 where other growth regimes are studied in more detail.

3 Transitional Dynamics

This section explores the transition dynamics of consumption and physical capital.

3.1 Dynamics of consumption

Rate of growth of consumption

$$g_C(z) = (\alpha z - \rho - \delta)/\sigma,$$

mimics the pattern of z along the transition. Since z is defined on the whole nonnegative real axis, it is possible for z_0 to fall short of $(\rho+\delta)/\alpha$. This implies $g_C < 0$ and consumption thus falls along the transition. Since z adjusts monotonically to $\widetilde{z}$, $z(t)$ eventually passes its threshold and consumption begins to rise. In this case consumption exhibits a nonmonotonic time profile. It firstly declines and reaches its minimum at

$$t_1 \equiv z^{-1}((\rho+\delta)/\alpha),$$

and then rises with an increasing rate.

The time path of consumption is different when $(\rho+\delta)/\alpha \leq z_0 < \widetilde{z}$. In this case, it grows at an increasing rate. Finally when $z_0 > \widetilde{z}$, consumption grows along the transition at a decreasing rate. In all cases because of sustainability of growth, i.e. condition (10), g_C eventually exceeds zero and approaches its long-run value on the balanced growth path, $\widetilde{g} > 0$. The findings are summarized as follows:

Proposition 1 *In the Lucas model, described by Eqs. (1)-(4), for $\alpha < \sigma$, the transition of consumption is governed through the following pattern:*

(i) If $z_0 < (\rho+\delta)/\alpha$, $C(t)$ falls on $0 \leq t \leq t_1$. It rises with an increasing rate afterwards. This results in a U-shaped path for consumption whose minimum equals $C(t_1)$. The stronger the consumption smoothing effect, i.e. larger σ, the shallower is the U-path.

(ii) If $(\rho+\delta)/\alpha < z_0 < \widetilde{z}$, C grows with an increasing rate along the off-balanced path.

(iii) If $z_0 > \widetilde{z}$, C rises with a decreasing rate along its transition path.

3.2 Dynamics of physical capital

The locus $\dot{\chi} = 0$, with a positive intercept, is flatter than the 45^0 line and crosses it at $z = (\rho + \delta)/\alpha$ which by (10) is less than $\widetilde{z}$. On the other hand, by (7), $\widetilde{\chi} < \widetilde{z}$. This gives $(\rho + \delta)/\alpha < \widetilde{\chi} < \widetilde{z}$.

For $0 \leq z < \widetilde{z}$, $\chi(z)$ lies between $\dot{\chi} = 0$ and $\chi = \widetilde{\chi}$. Moreover according to the direction of movements in the phase diagram (see figure 1), $\chi(z)$ is strictly increasing. This implies $(\rho + \delta)/\alpha < \widehat{z} < \widetilde{\chi} + \delta$,where $\widehat{z} = \arg\{\chi(z) = z - \delta\}$. In contrary to the Ramsey model, this indicates that there are two different thresholds corresponding to fall of consumption and physical capital in the Lucas framework.

Let $d(z)$ measures the distant between $\chi(z)$ and the $\dot{\chi} = 0$ locus, i.e. $d(z) \equiv \chi(z) - (1 - \alpha/\sigma)z + \delta - (\rho + \delta)/\sigma$, we have $d(z) > 0$ for $z < \widetilde{z}$ and $d(\widetilde{z}) = 0$ (see the upper panel of figure 1). Simple manipulation gives $g_K(z) = g_C(z) - d(z)$. This implies $g_K < g_C$ for $z < \widetilde{z}$.

3.3 Phases of transition

In general, when the average productivity of physical capital z is far short of its steady state $\widetilde{z}$, four episodes occur. First, when $0 < z(t) < (\rho + \delta)/\alpha$ both consumption and physical capital fall. During the second episode, when $(\rho + \delta)/\alpha < z(t) < \widehat{z}$, consumption rises but physical capital still decumulates, i.e. $g_K(z) < 0 < g_C(z)$. In the third phase, when $\widehat{z} < z(t) < \widetilde{z}$, both variables rise with different rates, i.e. $0 < g_K(z) < g_C(z)$. Finally, at the steady state they both grow at a common and constant rate, $g_K(\widetilde{z}) = g_C(\widetilde{z}) = \widetilde{g}$. The evolution of rates of growth of consumption and physical capital as functions of z along the stable arm is depicted in the lower panel of figure 1.

The transitional paths of consumption and physical capital, in the four episodes mentioned above, are depicted in the panels of figure 2. The first and third panels parallel those situations of the Ramsey model where an economy approaches the steady state from above and below respectively. The lower panel also addresses endogenous growth in the Lucas model which has no counterpart in the Ramsey model. The second panel, however, highlights one of the distinct features of transition of C and K in the Lucas model. In this stage, consumption rises while physical capital falls. We elaborate shortly the differences in the dynamics of consumption and physical capital in the Lucas and Ramsey model in detail.

Proposition 2 *In the Lucas model, described by Eqs. (1) - (4), where $\alpha < \sigma$, the followings hold.*

i) The rate of growth of consumption and physical capital adjust monotonously toward their common steady state $\widetilde{g}$. They change along their transition paths in the same direction and increase with the average productivity of physical capital. During transition, $g_K(z) \lesseqgtr g_C(z)$ for $z \lesseqgtr \widetilde{z}$ though as z approaches $\widetilde{z}$, they become closer to each other.

ii) For $z_0 < (\rho + \delta)/\alpha$, consumption and physical capital display U-shaped patterns on their transition paths. They fall for a finite period and then rise. In spite of the Ramsey model, here there are two different thresholds determining whether consumption and physical capital fall or rise during the adjustment process. The minimum of consumption always occurs before that of physical capital, i.e. $\arg\min\{C(t)\} < \arg\min\{K(t)\}$.

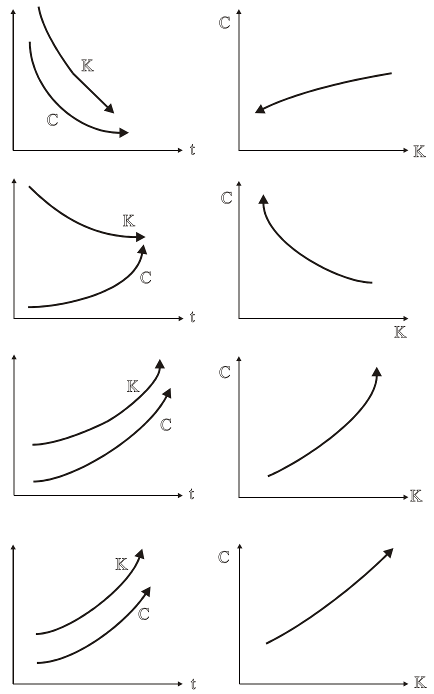

Figure 2: Phases of transition of consumption and physical capital

iii) According to the sequence of stages of transition, there exists a situation where a boost in consumption coincides with decumulation of physical capital.

The occurrence of a negative rate of growth on the transition depends on the size of the initial value of $z = (uH/K)^{1-\alpha}$ which is a measure of imbalance between physical and human capital relative to its steady state. The more scarce the level of skills with respect to the equipment and plant, the lower z, and it is more likely for z to fall behind the thresholds corresponding to C and K resulting in their U-shaped path during transition. Hence the falling of consumption and physical capital depends heavily on the extent of imbalance between the two sectors. Furthermore the less productive the technology of human capital accumulation, the longer is the falling period.

The likelihood of the occurrence of the falling period for an economy which is subject a the high ratio of physical to human capital depends on the magnitude of the effective discount rate $\rho + \delta$ relative to the productivity of education technology, B on one hand and curvature of the policy function $\chi(z)$ on the other. The higher $(\rho + \delta)/B$, consumption is more likely to go down during its transition, making the first stage longer. The flatter the policy function, $\chi(z)$ on the other hand, the farther is $\widehat{z}$ from its lower bound, $(\rho + \delta)/B$ and closer it is to its upper bound, $\widetilde{\chi}$ implying that the second period lasts longer. In addition, the closer $\widehat{z}$ to $\widetilde{z}$, the longer the third period lasts.

The range of benchmark values of parameters in the literature suggests that the above mentioned threshold might be close to $\widetilde{z}$. For example the corresponding figure in Mulligan and Sala-i-Martin(1993) and Barro and Sala-i-

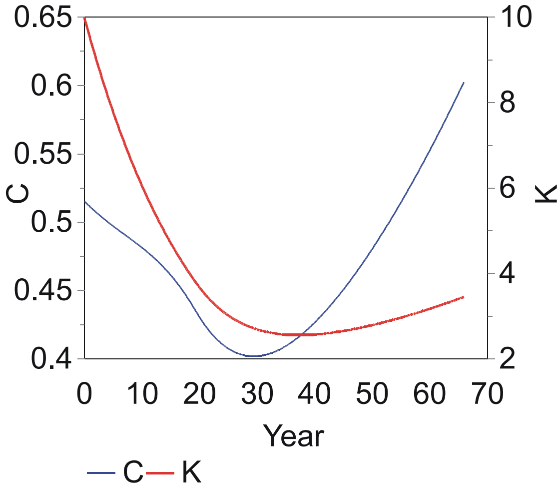

Figure 3: Time profile of consumption and physical capital

Martin(1995, sec. 5.2.2) is respectively 75 and 63.5 percent. This states that consumption and physical capital might fall even when the economy is quite close to the steady state.

3.4 Numerical simulation

We consult the time elimination method, introduced by Mulligan and Sala-i-Martin (1993), to solve the system of growth differential equations described by Eq. (6). For this purpose we apply the set of their base line parameters: $A = 1$, $B = 0.12$, $\alpha = 0.5$, $\delta = 0.05$, $\rho = 0.04$ and $\sigma = 2$. These give the steady state as $\widetilde{z} = 0.24$, $\widetilde{\chi} = 0.175$, $\widetilde{u} \cong 0.458$, and $\widetilde{g} = 1.5\%$.

We solve the system backwards by taking the steady state as the initial condition while by choosing eigenvector corresponding to the stable eigenvalue $\lambda = -0.12$, the solution traces the stable arm in (z, χ, u) space. The algorithm is described in appendix A. The results are depicted in figures 3 and 4.

Figure 3 refers to the time profiles of consumption and physical capital when the average productivity of capital is short of its steady state. The U-shaped path of consumption and physical capital is apparent from this figure. As one can observe, the falling period of C and K last more than 29 and 37 years respectively.

The first, second and third episodes occur in periods $[0, 30)$, $[30, 38)$, $[38, 67)$ respectively. These are depicted in the (C, K) space in figure 4 where the curve bends during the second stage of transition. Our numerical simulation shows that the findings are valid for a wide range of parameters as long as they are in their meaningful ranges and $\alpha < \sigma$.

3.5 Comparison with overaccumulation of capital in the Ramsey model

Here we briefly compare our findings with the Ramsey model where capital exceeds the modified golden rule and both consumption and capital decline monotonically toward their steady state.

The intuition behind falling are similar in the Lucas and Ramsey models, though they produce different transition paths. In both settings, according to the Euler equation, consumption falls when marginal productivity of (physical) capital is short of the effective discount rate. Falling of (physical) capital also, in both cases, is attributed to the case where consumption exceeds net output.

In the Ramsey model, falling of C and K coincide so that if capital exceeds the modified golden rule both consumption and capital decline monotonically toward their steady state. In the Lucas model instead, there are two different thresholds associated with falling of consumption and physical capital. Moreover steady state occurs where the *rate of growth* of the level variables, rather than their *level* themselves, are steady.

The cause of decline of C and K in the Lucas model is attributed to the high ratio of K to H which happens either due to the relative abundance of physical capital or the shortage of human capital. This measure of *sectorial imbalance* is more comprehensive than the mere overaccumulation of capital in the Ramsey model. Moreover, here the productivity of human capital accumulation is the key factor that determines, due to the intersectoral imbalances, how long the economy will stay in the falling period.

Finally, the time profile of consumption and physical capital are nonmonotonic here. They firstly fall and then gradually rise at an increasing rate. This happens in the presence of sustainability of growth in the steady state while there is no endogenous growth in the Ramsey model.

To make the comparison clearer, let us consider a Ramsey economy where labour, in efficiency units, grows at an exogenous rate. Along the transition, consumption and physical capital comove in the same direction monotonically toward their steady state paths. They both rise when the rate of return exceeds the effective discount rate, $\rho + \delta$ and fall when the reverse is the case. This replicates stages 1, 3 and 4 above but can not generate stage 2 in which the rise of consumption coincides with the decline of physical capital.

4 Exogenous and Paradoxical regimes

In the exogenous growth regime, i.e. when $\alpha = \sigma$, convergence of z_0 toward $\widetilde{z}$ is the only governing force of the dynamics of the economy. The $\dot{\chi} = 0$ locus, in this case, is horizontal meaning that, $\chi(z) = \widetilde{\chi} = (\rho + \delta)/\alpha - \delta$, for $z > 0$. This, by the third equation in (8) implies $u(z) = \widetilde{u}$ for all values of z too. The transition path, thus, is a horizontal curve parallel to the z axis:

$$(z\ ,\ \frac{\rho+\delta}{\alpha} - \delta\ ,\ 1 - \frac{B-\rho-\delta}{B\alpha} - \frac{\delta}{B}).$$

The Lucas model, in the exogenous regime replicates dynamics of the Ramsey model where human capital development acts like an exogenous technological

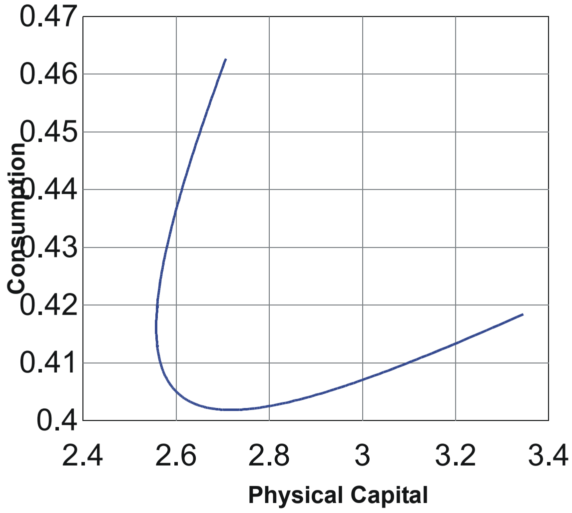

Figure 4: Evolution of consumption and physical capital in the (K, C) plane

progress. The production function (2), reduces in this case to

$$Y_t = a \exp\left[\widetilde{g}\,(1-\alpha)\,t\right] \cdot K_t^{\alpha}, \qquad a \equiv A(\widetilde{u}H_0)^{1-\alpha}.$$

The main difference with the Ramsey model, in this case, is that the rate of technological progress is determined by the optimal allocation of time between working and schooling.[5] This can be called an endogenous growth prototype of the Ramsey model.

Two thresholds for falling of C and K, in this case, coincide and the second stage of transition disappears. Since χ is constant, we have $g_K(z) = g_C(z)$ for all values of z. For $z_0 < (\rho+\delta)/\alpha$, consumption and physical capital still exhibit U-shaped profiles along their transition paths. In contrast with the normal regime, minimums of C and K now occur at the same time and $C(t)$ and $K(t)$ are parallel.

Using the same approach, one may conclude that in the paradoxical regime, i.e. where $\sigma < \alpha$, both $\chi(z)$ and $u(z)$ are decreasing. The phase diagram in the (z, χ) plane shows that $\chi(z)$ crosses the 45^0 line at a point at which $z < (\rho+\delta)/\alpha$. This means that the threshold for falling of physical capital, $\widehat{z}$, in this regime is less than that of consumption, $(\rho+\delta)/\alpha$. As a result, in the second stage falling of consumption coincides with rising of physical capital while other stages occur in a similar fashion with the normal regime.

The following proposition summarizes these findings

Proposition 3 *In the Lucas model, described by Eqs. (1)-(4), when the marginal*

[5] The model also differs in two other aspects from the Ramsey model. Consumption is limited to be proportional with physical capital along the whole adjustment period. The consumption smoothing effect is also weak.

productivity of physical capital is sufficiently low, four stages occur in transition of consumption and physical capital.

(i) In the first stage both consumption and physical capital fall. Duration of this stage, when $\alpha \leq \sigma$ is equal $t_1 = z^{-1}\left((\rho+\delta)/\alpha\right)$, while for $\sigma < \alpha$, this stage lasts $t_2 = z^{-1}\left(\arg \chi(z) = z - \delta\right)$.

(ii) In the second stage, if $\alpha < \sigma$, the rise of consumption coincides with the fall of physical capital whereas for $\sigma < \alpha$, consumption declines when physical capital rises. This stage does not exist if $\sigma = \alpha$. The second stage for $\alpha < \sigma$ terminates when $t = t_2$. In the case where $\alpha > \sigma$, it ends when $t = t_1$.

(iii) In the third stage, both consumption and physical capital rise. If $\alpha < \sigma$, consumption grows faster than physical capital while for $\sigma < \alpha$, the reverse is true. In the case where $\sigma = \alpha$, both variables grow at the same rate. Duration of this stage, for $\sigma \leq \alpha$ is equal $(t_2, +\infty)$ while for $\sigma < \alpha$, the time span of the third stage is equal $(t_1, +\infty)$.

(iv) Finally, in the fourth stage, both variables grow at their common long run rates.

5 The effect of human capital externality

Lucas suggests that the accumulation of human capital has an external effect on productivity of labour. An individual's productivity rises if others are more productive. He adds, therefore, an external effect for the economy's average level of human capital, $\overline{H}$ to the production function (2):

$$Y = AK^{\alpha}(uH)^{1-\alpha}\overline{H}^{\gamma}. \tag{13}$$

The over bar on H indicates that agents take this quantity as given in their decision. By introducing the externality effect, the competitive equilibrium departs from being necessarily Pareto optimal. For a social planner, H has the exponent $1 - \alpha + \gamma$ because besides its direct effect, she would take into account that human capital development raises output through externality. In the decentralized economy however, the representative agent regards $\overline{H}$ as a parameter beyond her decision.

It is the external effect, $\gamma > 0$ that distinguishes the social valuation of human capital accumulation from its private valuation. We consider these two cases separately in an exposition similar to that of Garcia-Castrillo and Sanso(2000).

5.1 The decentralized economy

Consider a competitive economy populated with a mass of households; each supplies one unit of labour inelastically at a skill level H. Labour can be employed in a fraction u in the product market and obtains a wage w per unit of skill that is taken as given. She also devotes a fraction $1 - u$ of time to education, improving her skill level in the form expressed in (4). The household is also endowed with a stock of assets K bearing the market interest rate r.

The income received in the form of labour and capital income is devoted to consumption, C and gross saving, $dK/dt + \delta K$. The budget constraint thus can be written as

$$\dot{K} = wHu + rK - C - \delta K, \tag{14}$$

where both physical capital and efficiency labour receive their marginal product:

$$w = (1 - \alpha)Y/(uH), \qquad \text{and} \qquad r = \alpha Y/K. \tag{15}$$

The problem the representative household faces is to maximize (1) subject to the restrictions (4) and (14) where $K(0)$ and $H(0)$ are given and the technology of aggregate production is (13). Moreover, since every household makes the same decision, consistency requires, after optimization, that $\overline{H} = H$.

The matrix of coefficient of the system of growth differential equation, (6) in this case is equal to (see appendix B)

$$M^e = \begin{bmatrix} \alpha - 1 & 0 & -B\gamma/\alpha \\ \alpha/\sigma - 1 & 1 & 0 \\ 0 & -1 & B(\alpha - \gamma)/\alpha \end{bmatrix},$$

where e stands for competitive equilibrium. This, in the absence of externality, i.e. if $\gamma = 0$, reduces to M in (6).

In comparison with our previous results in section 3, one can see that M^e is no longer triangular and the dynamics of the output-capital ratio is not self contained. One cannot, therefore, work out the dynamics of the output-capital ratio and consumption-capital ratio using the phase diagram in the (z, χ) plane. The off-balanced dynamics of consumption and physical capital are no longer tractable.

5.2 The socially optimal path

The optimal path from the social point of view is the solution adopted by a benevolent planner who takes all relevant information, including externality, into account. Handling the necessary conditions in a way analogous to that of the case of the decentralized economy, gives a triangular matrix of coefficient for system of growth differential equation (6) similar to M, except m_{33} is now replaced by $B + B\gamma/(1 - \alpha)$ which for $\gamma = 0$ reduces to M itself. (see appendix B)

Now in contrast to the decentralized economy, the dynamics of the output-capital ratio is self contained. In a similar way as section 3, one can now obtain the U-shaped path of consumption and physical capital.

Proposition 4 *In the presence of human capital externality, our findings in section 3 about the dynamics of consumption and physical capital, can only be validated in the centrally planned version of the model and are not extendable to the decentralized economy.*

Next section offers an immediate application for our findings.

6 Sustainability revisited

This section challenges the consistency of the two concepts of *endogenous growth* and *sustainable development.* In light of our findings in section 3, we examine how, in the coexistence of steady state and transitional dynamics, the two concepts may diverge.

In practice, sustainability may be simply defined as a non-declining level of well-being across time. This can be examined according to the *flow-based* or *stock-based* measures where real per capita consumption and physical capital are two well known indicators for the former and latter respectively; Hanley(2000).

In the view of modern growth theorists, sustainability of growth is the central question to which endogenous growth theory is addressed; Aghion and Howitt(1998, ch.5). There is a debate, however, among economists about the relation between *optimality* and *sustainability.* According to Anand and Sen(1995), "there is no general presumption that sustainability will be implied by optimality in models of intertemporal allocation."(p.5) Also "optimality and sustainability are logically distinct criteria of development. One can not be deduced from the other as a necessary consequence."(p.36)

Dasgupta(1994) on the other hand believes that sustainable development is almost exactly what has been analyzed for decades now in the literature on optimal growth theory.

A Lucas-type endogenous growth model, as explained in section 2, exhibits an optimal steady state in which growth is sustainable and output, consumption, physical and human capital all grow with a positive and common rate. This achievement reconciles two logically independent concepts of optimal growth and sustainable development.

In line with Solow(1970) who believes that "the steady state is not a bad place for the theory of growth to start, but may be a dangerous place for it to end."(p.7), we argue that the above picture is not complete as the transition stage has been overshadowed by the steady state. To understand the relation between optimality and sustainability, the transition path is as important as the steady state.

Our findings show that in the Lucas model, under certain circumstances, it is optimal for consumption and physical capital to display a negative rate of growth along the transition path. This means that although all level variables eventually grow at a common and positive rate, both consumption and physical capital might fall during their transition period implying that sustainability is violated during transition. Hence, although endogenous growth theory reconciles optimality and sustainability as two different criteria that a desirable path of development should meet, when transition is taken into account the two issues are still in conflict. This happens in a first best economy that is subject to an initial imbalance between its stock of machinery and equipment on one hand and its accumulated skills on the other hand. The main message, thus, is to point out the importance of the transition period in multi-sector endogenous growth models.

Optimal growth theory is an efficiency issue and the Maximum Principle as an established method, fulfils this task very well. Sustainability on the other hand is a matter of equity rather than efficiency. The former looks for an optimal path that maximizes the present value of instantaneous utility of consumption as a welfare criteria, while the latter is concerned with the non-declining path of consumption, income and physical capital.

Our discussion shows that the steady state analysis in the case where the rate of convergence is low and/or the economy is far away from its steady state may be misleading. Although, thanks to endogenous growth theory, two notions were reconciled in the steady state, there exist situations where optimality criteria departs from sustainability when the optimal path of consumption and physical capital falls over time.

The falling period might also take a long time if the productivity of skill acquisition is sufficiently low and if the stock of human capital is very scarce in relative terms. This happens in the framework of an endogenous growth model where the long-run growth is sustained forever implying that consumption and physical capital eventually rise along their transition paths.

7 Conclusion

The key assumption in the Lucas model, that education depends only on the pre-accumulated stock of human capital and the amount of time devoted to learning, identifies a cluster of multi-sector endogenous growth models where only one type of capital is used across other sector(s). Within this class we explore the transitional dynamics of consumption and physical capital. The results are as follows:

1. When human capital is relatively scarce, consumption and physical capital may exhibit U-shaped trends during their convergence paths toward their steady states. They firstly fall for a finite period and then rise toward their balanced growth paths.

2. The sequence of falling and rising of consumption and physical capital do not match. There exists situations where the rise of consumption coincides with the fall of capital and vice versa. This distinguishes the dynamics of the Lucas-type growth models from overaccumulation of capital in the Ramsey economy. Moreover, the sequence of stages in two models differ.

3. Rates of growth of consumption and physical capital, like the Ramsey model, move in the same direction during the transition, albeit their signs do not change at the same time.

4. Imposing the baseline parameters, one can observe that falling period does happen over a reasonably long period of time.

5. The existence of a falling stage in the transition paths of consumption and physical capital shows that in an economy that meets the sustainability condition, proposed in endogenous growth models, consumption and physical capital may fall along an optimal path violating the sustainabilty along the transition path.

6. These findings are applicable to multi-sector endogenous growth models that share the Lucas assumption. This includes R&D-based growth models so long as only one type of capital is used across both sectors. Established R&D-based endogenous growth models in both horizontal and vertical innovation versions meet this criteria. This includes the expanding variety model of Romer and the creative destruction model of Aghion and Howitt where the stock of human capital can be replaced by the variety of brands and the stock of social knowledge respectively. In addition, the fraction of effort devoted to education can be replaced by the amount of research employment in both models.

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A The algorithm for numerical simulation

This part explains an algorithm, based on the time elimination method, which is used in section 3.4 for simulating the off-balanced path of the Lucas model. The method firstly introduced by Mulligan and Sala-i-Martin(1993) into the field.

Consider a Lucas economy defined by Eqs. (1) - (4) with known parameters A, B, α, ρ, δ, σ. The off-balanced path of this economy is a sequence of points (z_i, χ_i, u_i) that are generated through the following algorithm.

1. Initialization:

   (a) Let $\epsilon > 0$ be a small deviation from the steady state[6] defined by (7), Δt the step size and $T \leq -2\ln(\widetilde{z}/\epsilon - 1)/\lambda$ the length of simulation.

   (b) Let $z_0 = \widetilde{z} - \epsilon$, $\chi_0 = \widetilde{\chi} - \epsilon v_0$ and $u_0 = \widetilde{u} - \epsilon w_0$ define the initial condition in the vicinity of the steady state where v_0 and w_0 are the second and third component of the stable eigenvector, λ defined in Eq. (12).

   (c) Set $i = 0$.

2. Going backward from the steady state, generate the next point on the saddle path as follow:

   (a) $\Delta z_i = (1-\alpha) z_i (\widetilde{z} - z_i)\Delta t$;

   (b) $z_{i+1} = z_i - \Delta z_i$;

   (c) $\chi_{i+1} = \chi_i - v_i \Delta z_i$;

   (d) $u_{i+1} = u_i - w_i \Delta z_i$,
   where

$$
\begin{aligned}
v_i &= \frac{(\alpha/\sigma - 1)z_i + \chi_i - \rho/\sigma + (1 - 1/\sigma)\delta}{(\alpha - 1)z_i - \lambda} \cdot \frac{\chi_i}{z_i}, \\
w_i &= \frac{-\lambda + Bu_i - \chi_i}{(\alpha - 1)z_i - \lambda} \cdot \frac{u_i}{z_i}.
\end{aligned}
$$

3. If $i \geq T$ algorithm ends. Otherwise set $i = i + 1$, Go to step 2.

The sequence $\{z_i, \chi_i, u_i\}_{i=1}^{T}$ presents the off balanced path of the economy for $z < \widetilde{z}$.

B Solution of the model with externality

This appendix gives solution of the Lucas growth model with externality presented in section 5. This also in the absence of externality, i.e. where $\gamma = 0$, produces the solution of the model in section 3.

The current value Hamiltonian of the model adopted by the central planner is given by

$$
J = \frac{C^{1-\sigma} - 1}{1 - \sigma} + \nu \left[AK^{\alpha}(uH)^{1-\alpha}\overline{H}^{\gamma} - C - \delta K \right] + \mu[B(1-u)H - \delta H],
$$

[6] Since z approaches asympthotically to $\widetilde{z}$, it never reaches it in practice. We thus displaces $\widetilde{z}$ by a small disturbance ϵ. For $\epsilon < 0$, the algorithm generates the right saddle path when $z > \widetilde{z}$.

where ν and μ are the costate variables. The term in the first set of brackets equals dK/dt, and the term in the second set of brackets equals dH/dt. The former in the competitive economy is replaced by $(r-\delta)K + wHu - C$ where r and w are defined in (15).

The first-order conditions, $\partial J/\partial C = 0$ and $\partial J/\partial u = 0$, lead respectively to

$$C^{-\sigma} = \nu, \tag{16}$$

and

$$\nu(1-\alpha)Y/u = \mu BH, \tag{17}$$

where in (16) the isoelastic form of the instant utility from (1) is taken into account. Condition (17) in the competitive economy is simply $\nu w = \mu B$.

Condition $d\nu/dt = \rho\nu - \partial J/\partial K$ implies

$$\dot{\nu}/\nu = \rho - \alpha z + \delta, \tag{18}$$

where the right hand side in the decentralized version is $-(r-\rho-\delta)$.

Condition $d\mu/dt = \rho\mu - \partial J/\partial H$ implies

$$\dot{\mu}/\mu = \rho - (1-\alpha+\gamma)Y/H - B(1-u) + \delta,$$

in which the consistency condition $H = \overline{H}$ has been imposed. In the competitive economy however, owing to lack of consideration of the human capital externality, the second term of the right hand side is wu.

The transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t}\nu_t K_t = 0, \qquad \lim_{t\to\infty} e^{-\rho t}\mu_t H_t = 0.$$

Differentiating Eq. (16) with respect to time gives, in conjunction with (18), the Euler equation for consumption growth

$$g_C = (\alpha z - \rho - \delta)/\sigma. \tag{19}$$

Moreover (3) gives the rate of growth of physical capital as,

$$g_K = z - \chi - \delta. \tag{20}$$

The growth rate of χ can then be determined from Eqs. (19) and (20) which is valid for both decentralized and centrally planned economies:

$$g_\chi = g_C - g_K = (\alpha/\sigma - 1)\,z + \chi - (\rho+\delta)\,/\sigma - \delta. \tag{21}$$

If one substitutes now for ν/μ from (17) into (18), the result is

$$\begin{aligned} \dot{\nu}^{o}/\nu &= -(B-\rho-\delta) - \gamma Bu/(1-\alpha), \\ \dot{\nu}^{e}/\nu &= -(B-\rho-\delta), \end{aligned}$$

where the superscript o and e refer to the optimal path and competitive equilibrium respectively. On the other hand differentiation with respect to time from (17) gives another expression for the shadow price of human capital as:

$$\dot{\nu}/\nu = \rho + \delta(1-\alpha) - \alpha\chi - \alpha g_u + (\gamma-\alpha)[B(1-u) - \delta].$$

These two expressions obtained for $\dot{\nu}/\nu$, give the rate of change of work effort along the optimal path and in the competitive equilibrium respectively as

$$g_u^o = -\lambda - \chi + Bu\,(1-\alpha+\gamma)\,/(1-\alpha), \tag{22}$$

$$g_u^e = -\lambda - \chi + Bu(\alpha-\gamma)/\alpha, \tag{23}$$

where $\lambda = -[B(1-\alpha+\gamma) - \delta\gamma]/\alpha$.

Differentiating Eq. (13) with respect to time, in conjunction with the above results, gives, after simplifying,

$$g_z^o = g_Y^o - g_K = -\lambda + (\alpha-1)z, \tag{24}$$

$$g_z^e = g_Y^e - g_K = -\lambda + (\alpha-1)z - Bu\gamma/\alpha. \tag{25}$$

Equations (24), (21) and (22) form a system of three growth differential equations in the variables z, χ and u, given the initial state variable $z(0)$. In the decentralized economy the first and third equations are replaced with (25) and (23).

Along a balanced growth path, by definition, rates of growth of C, K, Y and H are constant and u does not change. From (19), we conclude that z at the steady state is constant meaning that in the long run Y and K are proportional. This, in conjunction with (20), implies that at the steady state χ is also constant meaning that C and K grow in the long run at a common rate too. We call this balanced rate of growth $\widetilde{g} \equiv \widetilde{g_C} = \widetilde{g_K} = \widetilde{g_Y}$, where tilde refers to the steady state. If we differentiate Eq. (13) with respect to time and substitute from above, it results:

$$\widetilde{g} = (1+\gamma/(1-\alpha))\,\widetilde{g_H}. \tag{26}$$

The steady state of the centrally planned economy can be found readily by setting the three growth Eqs. (24), (21) and (22) to zero. This results

$$\begin{aligned}
\widetilde{z} &= [B(1-\alpha+\gamma) - \delta\gamma]/[\alpha(1-\alpha)], \\
\widetilde{\chi} &= B/\alpha - (B-\rho-\delta)/\sigma - \delta - (B-\delta)\,(1/\sigma - 1/\alpha)\,/(1-\alpha), \\
\widetilde{u} &= 1 - \{\rho - [\lambda - (1-\alpha)(\rho-\delta)]\,/\,[\sigma(1-\alpha+\gamma)]\}\,/B.
\end{aligned}$$

For the decentralized economy one should set the Eqs. (25), (21) and (23) to zero to obtain

$$\begin{aligned}
\widetilde{z} &= B/\alpha + [(B-\rho-\delta)\gamma/\alpha]/\,[\sigma(1-\alpha+\gamma) - \gamma]\,, \\
\widetilde{\chi} &= \widetilde{z} - \widetilde{g} - \delta, \\
\widetilde{u} &= 1 - [(1-\alpha)(B-\rho-\delta)/B]/\,[\sigma(1-\alpha+\gamma) - \gamma] - \delta/B.
\end{aligned}$$

The corresponding steady state growth rate of Y, C and K in the centrally planned and decentralized economy are:

$$\widetilde{g}^o = \frac{(B-\rho)(1-\alpha+\gamma)}{\sigma(1-\alpha)} - \frac{\delta}{\sigma}, \quad \text{and} \quad \widetilde{g}^e = \frac{B-\rho-\delta}{\sigma - \gamma/(1-\alpha+\gamma)}.$$

These two, taking into account (26), give the long run rate of human capital accumulation at a lower rate equal to

$$\widetilde{g}_H^o = \frac{(B-\rho-\delta)(1-\alpha)+\gamma(B-\rho)}{\sigma(1-\alpha+\gamma)}, \quad \text{and} \quad \widetilde{g}_H^e = \frac{(1-\alpha)(B-\rho-\delta)}{\sigma(1-\alpha+\gamma)-\gamma}.$$

The external effect therefore induces more rapid physical than human capital accumulation in such a way that overall the economy grows faster.

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