A Homotopy Approach to Solving Nonlinear Rational Expectation Problems

Mark J. Jensen
Department of Economics, Southern Illinois University
Carbondale, IL 62901

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Abstract

Many numerical methods have been developed in an attempt to find solutions to nonlinear rational expectations models. Because these algorithms are numerical in nature, they rely heavily on computing power and take sizeable cycles to solve. In this paper we present a numerical tool known as homotopy theory that can be applied to these methods. Homotopy theory reduces the computing time associated with an iterative algorithm by using a rational expectation problem with known solutions and transforming it into the problem at hand. If this transformation is preformed slowly, homotopy theory will also help the the global convergence properties of the numerical algorithm. We apply homotopy theory to Den Haan and Marcet's Parameterized Expectation Approach to show how homotopies improves the computing speed and global convergence properties of this algorithm.

1 Introduction

In the last five or so years macroeconomists have developed a number of nonlinear rational expectation models that are more realistic in their description of economic behavior. As a result, these rational expectation models have become very complex and can not be solved in a analytical manner. Rather, solutions to these complex rational expectation models require numerical techniques to provide solutions. Many numerical methods have been developed in an attempt to find solutions to these nonlinear rational expectations models that agree with theory and empirical findings.[1] Because all of these algorithms are numerical in nature, even for the simplest nonlinear rational expectation model they rely heavily on computing power and take sizeable cycles to solve.

In this paper we present a numerical tool that can be applied to these methods. Known as homotopy theory, this tool aids in reducing the computing time associated with an iterative algorithm by taking a rational expectation problem with known solutions and transforming it into the problem at hand. If this transformation is done slowly, homotopy theory also reduces the chance that the numerical algorithm will lose its way while iterating towards a solution.

We apply homotopy theory to Den Haan and Marcet's (1991) Parameterized Expectation Approach (PE). The PE approach was chosen because it lacks a gradient-like update. This shortcoming causes the PE approach to have very poor global convergence properties and hence, a high likelihood of explosive behavior. Homotopy theory reduces the chance of the PE method exploding, thus, improving the algorithm's global convergence properties and its computing time.

The nonlinear rational expectation problem to which we apply the PE method and homotopy theory is a representative consumer's stochastic decision problem over monetary and consumption goods. This optimization problem originated from Barnett, Hinich and Yue's (1991) work on monetary aggregation under risk. As we will see, this model consists of a representative agent who maximizes her additively time separable utility function over a monetary aggregate and an aggregate consumption good, subject to an intertemporal budget constraint. This dynamic optimization problem is a nonlinear model that maps endogenous and exogenous state variables into endogenous control variables.

The paper is organized as follows. Section 2 presents the representative agent's prefer-

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[1] See Taylor and Uhlig (1990) for a extensive analysis of the strength and weaknesses of the available algorithms.

ences and her optimization problem. In Section 3 we give an idea of how to use homotopy theory and the PE method are used to solve Section 2's nonlinear rational expectation problem. Lastly, Section 4 reports our findings on using homotopies with the PE approach to solve the model described in Section 2.

2 Structural Model

2.1 Consumer Demand for Money

To show the ability of homotopy theory we use the optimization problem of Barnett, Hinich and Yue (1991) where a consumer maximizes her intertemporal utility over a quantity aggregate for consumption goods and a quantity aggregate for monetary goods. The monetary aggregate is comprised of three monetary assets, whereas the consumption aggregate is viewed as a scalar.

The individual consumer is assumed to maximize her utility over the finite planning horizon $t, t+1, t+2, \ldots, t+T$, where t is the beginning time period and $t+T$ is the terminal time period. The representative consumer's utility function has the form

$$\begin{aligned} V(M(\boldsymbol{m}_t)^{1-\beta}X_t^{\beta}) \quad &+ \quad E_t\left[\sum_{s=t+1}^{t+T-1}\rho^{s-t}V(M(\boldsymbol{m}_s)^{1-\beta}X_s^{\beta})\right. \\ &+ \quad \left.\rho^T V_T(M(\boldsymbol{m}_{t+T})^{1-\beta}X_{t+T}^{\beta}, A_{t+T})\right] \end{aligned} \tag{1}$$

where V is the constant relative risk aversion utility function

$$V(M(\boldsymbol{m}_s), X_s) = \frac{1}{\sigma}\left[X_s^{\beta}M_s^{1-\beta}\right]^{\sigma} \tag{2}$$

with $\sigma \in (-\infty, 0) \cup (0, 1)$, X_s and M_s are respectively the consumption good and monetary good quantity aggregates, and $\boldsymbol{m}_s$ is a 3×1 quantity vector of monetary assets. If $\sigma \to 0$ the consumer's utility function is $V(M_s, X_s) = \ln(X_s^{\beta}M_s^{1-\beta})$. In Eq. (1) the allocation between the two aggregates is determined by β which upon closer observation is the parameter of a subnested Cobb-Douglas utility function, hence, $\beta \in (0, 1)$. The parameter ρ represents the agent's subjective discount factor with the restriction $\rho \in (0, 1)$.

The variable A_s represents the quantity of the benchmark asset that the consumer plans on holding in period s. Because the yield on A_s only serves to transfer wealth across time periods and does not yield liquidity nor any other services during the current period, the benchmark asset provides nothing to the consumer, and hence, only enters her utility function in the final time period of the planning horizon. Furthermore, the benchmark asset's period yield is defined to contain all the premiums of the market for foregoing the services provided by a monetary asset. Hence, the probability of the benchmark asset's yield exceeding the yields of all the other assets is nonzero, otherwise there would be no reason for the agent to hold the benchmark asset.

Lastly, the monetary aggregate, $M(\boldsymbol{m}_s)$, is defined as the CES function

$$M(\boldsymbol{m}_s) = \left(\sum_{i=1}^{3} \delta_i m_{is}^{\alpha} \right)^{1/\alpha} \tag{3}$$

with $\sum_{i=1}^{3} \delta_i = 1$, and $\alpha \in (0, 1]$. Even though the theoretical monetary aggregate is an explicit function of the individual monetary assets, we drop the function's arguments to simplify the notation and represent the monetary aggregate as M_s. The monetary aggregate will equal the Cobb-Douglas aggregator function if $\alpha \to 0$ and the linear aggregator function if α equals one.

2.2 Optimization Problem

The representative consumer's dynamic optimization problem consists of choosing the deterministic point $(\boldsymbol{m}_t', X_t, A_t)$ and the stochastic process $(\boldsymbol{m}_s', X_s, A_s)$ for $s = t+1, t+2, \ldots, t+T$ that maximizes Eq. (1) subject to the budget constraints

$$I_s \geq -\left\{ \sum_{i=1}^{3} \left[(1 + r_{i,s-1}) p_{s-1}^{*} m_{i,s-1} - p_s^{*} m_{i,s} \right] + (1 + R_{s-1}) p_{s-1}^{*} A_{s-1} - p_s^{*} A_s - p_s^{*} X_s \right\} \tag{4}$$

for $s = t, t+1, \ldots, t+T$. The budget constraint is comprised of the expected nominal holding period yields of the monetary assets, r_{is}, for $i = 1, 2, 3$, and the expected one-period holding yield on the benchmark asset during period s, R_s. Under the assumption of rational expectation the distribution of each asset's yield is known to the consumer. However, since the payment received on a interest accruing good does not occur until the end of the period the consumer will not know the actual value of either the r_{is}'s nor R_s during period s. On the other hand, the consumption good's price aggregate (or true cost of living index) p_s^{*} is determined and fully known by the representative agent at the beginning of period s. I_s represents the sum of all other sources of income during period s.

As derived by Barnett, Hinich and Yue (1991), this dynamic optimization problem has the following first-order necessary condition Euler equations

$$X_s^{\sigma\beta - 1} M_s^{\sigma(\beta - 1)} = E_s \left[\left(\rho \frac{p_s^{*}}{p_{s+1}^{*}} (1 + R_s) \right)^{-1} X_{s+1}^{1 - \sigma\beta} M_{s+1}^{\sigma(\beta - 1)} \right] \tag{5}$$

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{1s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_1}\frac{\beta\rho}{1-\beta}\frac{p_s^* R_s}{p_{s+1}^*}\right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right] \tag{6}$$

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{2s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_2}\frac{\beta\rho}{1-\beta}\frac{p_s^* (R_s - r_{2s})}{p_{s+1}^*}\right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right] \tag{7}$$

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{3s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_3}\frac{\beta\rho}{1-\beta}\frac{p_s^* (R_s - r_{3s})}{p_{s+1}^*}\right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right] \tag{8}$$

In the above dynamic programming problem, the consumer faces the exogenous stochastic state vector $\boldsymbol{\phi}_s = (R_{s-1}, r_{2,s-1}, r_{3,s-1}, p_s^*/p_{s-1}^*, I_s/I_{s-1})'$, along with the endogenous state vector $\boldsymbol{m}_{s-1}$. Together they define the state vector $\boldsymbol{\sigma}_s' = (\boldsymbol{m}_{s-1}', \boldsymbol{\phi}_s')$, while the representative consumer's control vector for the optimization problem is $\boldsymbol{z}_s' = (\boldsymbol{m}_s', X_s)$.

The stochastic process $\{\boldsymbol{\phi}_s\}$ is assumed to behave as two independent Markovian processes that have the behavior

$$\boldsymbol{\phi}_{1s} = \boldsymbol{a}_1 + \boldsymbol{A}_1 \boldsymbol{\phi}_{1,s-1} + \boldsymbol{u}_{1s} \tag{9}$$

$$\ln(\boldsymbol{\phi}_{2s}) = \boldsymbol{a}_2 + \boldsymbol{A}_2 \ln(\boldsymbol{\phi}_{2,s-1}) + \boldsymbol{u}_{2s} \tag{10}$$

where $\boldsymbol{\phi}_{1s} = (R_{s-1}, r_{2,s-1}, r_{3,s-1})'$ and $\boldsymbol{\phi}_{2s} = (p_s^*/p_{s-1}^*, I_s/I_{s-1})'$.[2] $\boldsymbol{u}_{1s}$ and $\boldsymbol{u}_{2s}$ are both distributed $i.i.d.\ \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Omega}_i)$, for $i = 1, 2$, with $\boldsymbol{A}_1$ and $\boldsymbol{\Omega}_1$ being 3×3 matrices, whereas $\boldsymbol{A}_2$ and $\boldsymbol{\Omega}_2$ are 2×2 matrices. The processes' intercept terms $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are 3×1 and 2×1 vectors, respectively.

Each Markovian process is a first-order vector autoregressive process (VAR(1)), the only difference between them being the natural logarithmic transformation of $\boldsymbol{\phi}_{2s}$. This transformation insures that the process is stationary. Since $\boldsymbol{\phi}_{1s}$ is only comprised of interest rates there is no reason to transform this process.

To insure that the benchmark asset's yield exceeds r_{2s} and r_{3s} for all s, R_s is generated in the following manner

$$R_s = \begin{cases} \max(r_{2s}, r_{3s}) + \frac{1}{4}\max(r_{2s}, r_{3s}) + u_{1s_1}, & \text{if } \frac{(R_s - \max(r_{2s}, r_{3s}))}{R_s} \leq 0.05 \\ R_s, & \text{otherwise} \end{cases}$$

where u_{1s_1} is the first component of $\boldsymbol{u}_{1s}$.

We have now completed the construction of the dynamic optimization problem. The representative agent makes her choice of the deterministic point $\boldsymbol{z}_t$ and the stochastic points

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[2] If desired, a more complex stochastic processes for the exogenous state process can be used.

$\boldsymbol{z}_s$ for $s = t+1, t+2, \ldots, t+T$ for a given parameter vector describing the economy. If we define the economy's parameter vector as

$$\boldsymbol{\lambda} = (\rho, \sigma, \alpha, \beta, \delta_1, \delta_2, \delta_3, \boldsymbol{a}_1', \boldsymbol{a}_2', vec(\boldsymbol{A}_1)', vec(\boldsymbol{A}_2)', vec(\boldsymbol{\Omega}_1)', vec(\boldsymbol{\Omega}_2)')$$

then for each value of $\boldsymbol{\lambda}$ the above optimization problem defines a nonlinear mapping from the exogenous state process $\{\boldsymbol{\phi}_s\}$ to $\{\boldsymbol{z}_s\}$.

3 Solving the Optimization Problem

3.1 Parameterized Expectation Approach

Although there has been a large demand for numerical methods that solve a dynamic optimization problem like that found in Section 2 only recently have such algorithms been devised.[3] Of those currently available to economists the Parameterized Expectation Approach (PE) of Den Haan and Marcet (1990) has performed well in head-to-head tests with other algorithms. In addition, the PE approach has been applied to a number of different economic areas, including growth models (den Haan and Marcet (1990)), asset markets with heterogeneous agents (Ketterer and Marcet (1989), Marcet and Singleton (1990)), and monetary economies (den Haan (1990a, 1990b), Marshall (1992)).

The PE method simply approximates the expectation operators found in the Euler equations (5)-(8) by parameterizing them with a basis function that spans the set of expectation operators. The basis function is a globally flexible functional form who's arguments are the state variables $\boldsymbol{\sigma}_s$.[4] In this paper we choose to use a first-order polynomial to approximate the expectation operators. One then iterates over the parameters of these flexible functions until a convergence criterion is met.

Intuitively, each iteration of the PE method can be viewed as a nonlinear least-square learning behavior by the consumer.[5] Once learning no longer occurs, the representative agent selects the stochastic solutions $\{\boldsymbol{z}_s\}$ that satisfies the Euler equations, given her learned prediction of the expectation operators. Hence, it can be argued that the numerical solution found with the PE approach is an equilibrium for a representative agent restricted to the learning associated with a specific globally flexible functional form.

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[3] See Taylor and Uhlig (1990) for excellent source on the available methods and a comparison on their performance record.

[4] See Barnett and Jonas (1982), Barnett and Yue (1988), and Gallant (1982) for the properties and examples of globally flexible functions.

[5] See Marcet and Sargent (1989a, 1989b) for examples of a linear least-square learning model that have a locally stable equilibrium.

To apply the PE method to the optimization problem in Section 2, we write the system of Euler equations (5)-(8) as

$$\Phi(\boldsymbol{z}_s) = E\left[\Gamma(\boldsymbol{z}_{s+1}, \boldsymbol{\phi}_{s+1}) | \boldsymbol{\sigma}_s\right] \tag{11}$$

where $\Phi : \Re^4 \rightarrow \Re^4$ and $\Gamma : \Re^9 \rightarrow \Re^4$. By defining each vector component of Γ's range as $g_i(\boldsymbol{z}_{s+1}, \boldsymbol{\phi}_{s+1})$, for $i = 1, 2, 3, 4$, we can write the PE parameterization of each Euler equation's expectation operator as

$$E[g_i(\boldsymbol{z}_{s+1}, \boldsymbol{\phi}_{s+1}) | \boldsymbol{\sigma}_s] = \exp[poly(\tilde{\boldsymbol{\sigma}}_s, \boldsymbol{v}_i)] \qquad i = 1, 2, 3, 4$$

where $\tilde{\boldsymbol{\sigma}}'_s = (1, \boldsymbol{\sigma}'_s)$, and the $\boldsymbol{v}_i$'s are the polynomial's coefficient vector.

The first step in the PE approach is to generate a single realization of the random variables $\{\boldsymbol{u}_{1s}\}$ and $\{\boldsymbol{u}_{2s}\}$ from their known distributions. These series along with the two VAR(1) functions, (9) and (10), are then used to generate the stochastic exogenous state processes, $\{\boldsymbol{\phi}_{1s}\}$ and $\{\boldsymbol{\phi}_{2s}\}$. To initiate the two Markovian processes, we chose to let $\boldsymbol{\phi}_{10} = (0.098, 0.039, 0.071)'$, the sample mean of the interest rates from an empirical data source, and $\boldsymbol{\phi}_{20} = \mathbf{1}'$.[6]

To begin the PE's iteration process the initial vector values are selected for each of the polynomial's coefficients, which we define as $\boldsymbol{v}_i^0$ for $i = 1, 2, 3, 4$. Like all numerical algorithms the choice of the $\boldsymbol{v}_i^0$'s is critical, but even more so with the PE approach since each iteration does not calculate a directional update. Hence, the PE algorithm only has good local convergence properties and thus, considerable attention should be made in selecting $\boldsymbol{v}_i^0$. Hence, the importance of homotopy theory in insuring that the initial coefficients are within the set which converges to a solution.

After the $\boldsymbol{v}_i^0$'s are chosen, each polynomial, $\exp[poly(\tilde{\boldsymbol{\sigma}}_s^0, \boldsymbol{v}_i^0)]$, is calculated and substituted into Eq. (11) for $E[\Gamma(\boldsymbol{z}_{s+1}, \boldsymbol{\phi}_{s+1}) | \boldsymbol{\sigma}_s]$. Eq. (11) is a nonlinear system of equations which can be solved explicitly for $\{\boldsymbol{z}_s\}$ for a special limiting case within the parameter space, but for most values of $\boldsymbol{\lambda}$ Eq. (12) is an implicit system of equations that requires a Newton-like algorithm to solve for $\{\boldsymbol{z}_s^1\}$.[7]

Homotopy theory is very useful in providing initial guesses for the Newton-like algorithm which increases the likelihood of convergence to the correct solution. In these type of

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[6] To reduce the dependence of the exogenous state variables on these initial values we exclude the first five periods from the simulation.

[7] In our actual calculations we use a modified Powell hybrid algorithm based on the MINPACK subroutine *HYBRD1* to solve the implicit system of equations.

situations, homotopy theory suggests first solving the optimization problem for the special parameter case where Eq. (12) can be explicitly solved for $\boldsymbol{z}_s$. The PE solution to the optimization problem for that value of $\boldsymbol{\lambda}$ provides us with a reasonably good choice for the starting values of the Newton-like algorithm when the optimization problem has a $\boldsymbol{\lambda}$ close to the special case.

Once $\{\boldsymbol{z}_s^1\}$ is calculated, the solutions are used to calculate $g_i(\boldsymbol{z}_{s+1}^1, \boldsymbol{\phi}_{s+1})$, for $i = 1, 2, 3, 4$. Each of these values are regressed on a linearized version of $\exp[poly(\tilde{\boldsymbol{\sigma}}_s^1, \boldsymbol{v}_i)]$. This linearization is performed around $\boldsymbol{v}_i^0$, which allows us to use ordinary least-squares rather than a nonlinear regression at each step of the iteration.

Because of the instability involved in the above iteration process for complex optimization problems, Den Haan and Marcet (1990) suggest that the updated coefficients for the polynomials be a convex combination of the estimated and previous iterations coefficients. In other words, the updated coefficients are calculated as

$$\boldsymbol{v}_i^1 = \eta \boldsymbol{b}_i^1 + (1 - \eta)\boldsymbol{v}_i^0 \tag{12}$$

where $\eta \in [0, 1]$ and $\boldsymbol{b}_i^1$ is the estimated coefficients vector found in the previous paragraph. The convex combination of the coefficient estimates has a stabilizing affect on the PE's iterations for models that are normally explosive. By choosing smaller values of η the updating of the polynomial's coefficients is more gradual and hence, more resistant to an explosive path.

The above steps are then repeated until the solutions $\{\boldsymbol{z}_s\}$ converge. We use the convergence criterion recommended by Bansal et. al. (1992) which is

$$\max_i \max_s \left| \frac{(z_{is}^k - z_{is}^{k-1})}{(z_{is}^{k-1} + \epsilon)} \right| \leq \xi \tag{13}$$

where ϵ and ξ are small positive number, and k is an index for the number of the iterations performed. Den Haan and Marcet (1990) originally suggested that the estimated coefficients vectors, $\boldsymbol{v}_i^k$, be tested for convergence. However, since the optimization problem contained in this paper has eight state variables that are possibly correlated, testing convergence with the coefficients of the polynomials would most likely suffer from the oscillating nature associated with the estimated parameters of a multicollinear model.[8]

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[8] Later we explain and use Marshall's (1992) sum of squared residuals convergence criterion.

3.2 Homotopy Theory

Homotopy theory although a very simple concept is a very powerful tool in finding the solution to a complex fixed point algorithm like the PE approach. The basic idea behind homotopy theory is to slowly move from a simple case, where the solution is known, to the desired case where the solutions is difficult to solve for and unknown. This simple idea of going from a known to a unknown solution enables many previously unstable and explosive algorithms to become well behaved numerical routines that quickly converge to a solution. To explain homotopy theory we apply it to the nonlinear system of equations (11).[9]

Suppose the vector,

$$\begin{aligned} < \exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}^k)] > &= (\exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}_1^k)], \exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}_2^k)], \\ &\quad \exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}_3^k)], \exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}_4^k)])' \end{aligned}$$

has been substituted for $E[\Gamma(\boldsymbol{z}_{s+1}^k, \boldsymbol{\phi}_{s+1})|\boldsymbol{\sigma}_s^k]$, as we instructed in Subsection 3.1, and subtracted from both sides so that the system of equations found in Eq. (11) can be written as

$$F(\boldsymbol{z}_s, \tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}^k) = \mathbf{0} \tag{14}$$

where $F : \Re^{49} \to \Re^4$ and $\boldsymbol{v}^k = (\boldsymbol{v}_1^k, \boldsymbol{v}_2^k, \boldsymbol{v}_3^k, \boldsymbol{v}_4^k)'$

We desire to solve Eq. (14) for $\boldsymbol{z}_s$ but we lack a good starting value for the Newton-like algorithm. However, we do know that $\boldsymbol{z}_s^k$ is the solution to Eq. (14) when the polynomial's coefficient vector equals $\boldsymbol{v}^{k-1}$, i.e. $F(\boldsymbol{z}_s^k, \tilde{\boldsymbol{\sigma}}_s^{k-1}, \boldsymbol{v}^{k-1}) = \mathbf{0}$. Using this information, we define $H : \Re^4 \times [0,1] \to \Re^4$, to be the homotopy function

$$H(\boldsymbol{z}_s, \tau) \;=\; \Phi(\boldsymbol{z}_s) - \Big\{(1-\tau) < \exp[\tilde{\boldsymbol{\sigma}}_s^{k-1}, \boldsymbol{v}^{k-1}] > +\tau < \exp[\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}^k] > \Big\}.$$

It follows that $H(\boldsymbol{z}_s^k, 0) = \mathbf{0}$, and $H(\boldsymbol{z}_s^{k+1}, 1) = \mathbf{0}$, where $\boldsymbol{z}_s^{k+1}$ is the solution to Eq. (14). Hence, the homotopy function, H, provides us with a function $\boldsymbol{z}_s(\tau)$ that satisfies $H(\boldsymbol{z}_s(\tau), \tau) = \mathbf{0}$ for all $\tau \in [0,1]$. By gradually moving τ from zero to one, $\boldsymbol{z}_s(\tau)$ maps out the path to the solution $\boldsymbol{z}_s^{k+1}$.

In our nonlinear rational expectation problem we not only use homotopy theory to solve $F(\boldsymbol{z}_s, \tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}^k) = \mathbf{0}$, we also use it to determine the initial values for the coefficient vector $\boldsymbol{v}^0$. If homotopy theory is to be useful in providing values for the initial coefficient

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[9] See Garcia and Zangwill (1981) for a more extensive study on homotopy theory being applied to fixed point problems.

vector, a solution to the nonlinear rational expectation problem must be know for some $\boldsymbol{\lambda}$. Unfortunately, we initially do not know the solution to our optimization problem for any value of $\boldsymbol{\lambda}$. To obtain an initial solution, we desire to find a $\boldsymbol{\lambda}_0$ that will simplify our optimization problem and enable the PE algorithm to converge. We choose the limiting case where $\sigma \to 0$ and $\alpha \to 0$ since this value of $\boldsymbol{\lambda}$ causes the Euler equations (5)-(8) to equal

$$\begin{aligned} X_s &= E_s\left[\left(\rho\frac{p_s^*}{p_{s+1}^*}(1+R_s)\right)^{-1}X_{s+1}\right]^{-1} & (15)\\ m_{1s} &= E_s\left[\left(\frac{1}{\delta_1}\frac{\beta\rho}{1-\beta}\frac{p_s^*R_s}{p_{s+1}^*}\right)^{-1}X_{s+1}\right] & (16)\\ m_{2s} &= E_s\left[\left(\frac{1}{\delta_2}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s-r_{2s})}{p_{s+1}^*}\right)^{-1}X_{s+1}\right] & (17)\\ m_{3s} &= E_s\left[\left(\frac{1}{\delta_3}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s-r_{3s})}{p_{s+1}^*}\right)^{-1}X_{s+1}\right]. & (18)\end{aligned}$$

Hence, in this limiting case the Φ function of Eq. (11) is the identity function. The homotopy function, $H(\boldsymbol{z}_s, 0)$, is formed by subtracting the RHS of (15)-(18) from the LHS.

The reader will notice that under this limiting parameter vector solving for $\boldsymbol{z}_s$ in Eq. (14) no longer requires a Newton-like numerical procedure. Rather, we are able to explicitly solve for the vector $\boldsymbol{z}_s^{k+1}$ by setting the solutions equal to the parameterized value of the expectation operator, $< \exp[poly(\tilde{\boldsymbol{\sigma}}_s^k, \boldsymbol{v}^k)] >$.

After convergence is reached with the PE algorithm for the parameter vector $\boldsymbol{\lambda}_0$, homotopy theory tells us to use the polynomial's coefficient vector, $\boldsymbol{v}^*$, from the PE solution as the starting coefficients $\boldsymbol{v}^0$ and the PE solutions $\{\boldsymbol{z}_s^*\}$ as initial guesses for the Newton-like algorithm when the parameter vector equals $(1-\tau)\boldsymbol{\lambda}_0 + \tau\boldsymbol{\lambda}^*$, for $\tau \in (0,1]$. Use of the PE solutions, $\boldsymbol{v}^*$ and $\{\boldsymbol{z}_s^*\}$, as the starting values to the PE algorithm for larger and larger τ continues until the desired combination of parameters, $\boldsymbol{\lambda}^*$, are reached or the PE algorithm fails to converge.

If the PE approach fails to converge, the homotopy function, H, is redefined so that $H(\boldsymbol{z}_s, 0) = \mathbf{0}$ for the most recently solved rational expectation model. A new $\boldsymbol{\lambda}_0$ is defined and $(1-\tau)\boldsymbol{\lambda}_0 + \tau\boldsymbol{\lambda}^*$ is again used with τ approaching 1.

4 Simulation Results

In this section we present our findings on using homotopy theory with the PE algorithm to solve the dynamic optimization problem in Eq. (1). All the calculations and algorithms were programmed in Fortan 77 and compiled and executed on a SPARC 10 workstation.[10] We experimented with different convergence levels and discovered that the strictest convergence criterion for which the PE approach would consistently converge under 40,000 iterations was $\xi = 0.03$. The number of iterations required for the PE algorithm to converge varied with the value of the parameter vector, but when the PE algorithm converged it usually did so within thirty minutes. In addition, when homotopy theory was used to provide the PE with initial values convergence took less than a few hundred iterations.

Convergence of the PE approach does not guarantee that the calculated solutions are the 'true' solutions of the optimization problem. Because the PE approach is a numerical algorithm it can only provide an approximation of the 'true' solution. Hence, to determine if the approximation is close to the 'true' solution we employ the Den Haan, Marcet test statistic (DHM-stat) [Den Haan and Marcet (1994) and Taylor and Uhlig (1990)]. The DHM-stat provides a test of the theoretical martingale property $E[\boldsymbol{\nu}_s \otimes \boldsymbol{h}_{sij}] = 0$ where $\boldsymbol{\nu}_s$ is the 4×1 residual vector from the Euler equations and $\boldsymbol{h}_{sij} = \boldsymbol{\phi}^i_{s-j}$. If Hansen's (1982) regularity conditions hold and the approximation is an exact solution satisfying the above martingale property then the DHM-stat

$$TBA^{-1}B \tag{19}$$

where $B = 1/T \sum_{s=t}^{t+T} [\boldsymbol{\nu}_s \otimes \boldsymbol{h}_{sij}]$ and $A = 1/T \sum_{s=t}^{t+T} [\boldsymbol{\nu}_s \otimes \boldsymbol{h}_{sij}][\boldsymbol{\nu}_s \otimes \boldsymbol{h}_{sij}]'$, will be distributed in law χ^2 with 20 degrees of freedom. In the following results we have set the DHM-stat's i and j equal to one.

In each case a first-order polynomial function was used to approximate the expectation operator. Thus, each PE algorithm solution is a first-order approximation to the 'true' solution. We attempted to use a second-order polynomial, both with and without cross-variable terms, but the PE approach became explosive after a few iterations. This volatile behavior can be traced back to the model's large number of state variables and how quickly the number of estimated coefficients goes from 9 to 65 coefficients.

Except for α and σ, table 1 lists the values for $\boldsymbol{\lambda}$ used in the PE algorithm. To insure

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[10] I am deeply indebted to Bob Hussey for providing and explaining the initial coding that I altered to fit this paper's model.

that $\boldsymbol{\lambda}$ accurately reflects the empirical world we set the parameters equal to the GMM estimates found by Barnett, Hinich and Yue (1991). The parameters from the Markovian processes (9)-(10) equal the estimated parameters from the VAR(1)s using monthly data from January 1960 to December 1990. Lastly, in table 1 the parameter values $H_{i_{jk}}$ for $i = 1, 2$, and $j, k = 1, 2, 3$, represent the Cholesky factorization of the random disturbances covariance matrix, i.e. $H_i H_i^{'} = \boldsymbol{\Omega}_i$.

To estimate the parameters of Eq. (10) we used the Consumer Price Index for p_s^* and nominal GNP for I_s. The interest rate data for Eq. (9)'s regression was supplied to us by the St. Louis Federal Reserve Bank. Each investment's rate of return was adjusted to a common one-month maturity with Farr and Johnson's (1985) yield curve adjustment to eliminate liquidity premiums associated with longer maturity investments. In addition, the investment's interest rates were adjusted to an annualized one-month yield on a bond interest basis (365 day) as opposed to a bank basis (360 day). For the regression of Eq. (9) the benchmark asset's rate of return was measured by

$$R_s = \max[r_{BAA}, (r_{is},\ i = 1, 2, 3)]$$

where r_{BAA} is the rate on Moody's BAA corporate bond, r_{2s} was measured by the average of the following investment returns

- RMMDAC: Rate paid on Money Market Deposit Accounts at commercial banks
- RMMDAT: Rate paid on Money Market Deposit Accounts at thrift institutions
- RSDCB: Rate on savings deposits less RMMDAC
- RSDSL: Rate on savings deposits at FDIC-Insured savings banks
- RSNOWC: Rate paid on Super NOWs at commercial banks
- RSNOWT: Rate paid on Super NOWs at thrift institutions
- RONRP: Rate paid on overnight dealer financing in the repurchase market
- RONED: Rate paid on overnight eurodollars from London
- RMMMF: Average yield on Money Market Mutual Funds

and r_{3s} was the average of the returns

- RLTDCB: Rate on large time deposits at commercial banks
- RSTDTH: Rate paid on small time deposits and retail repurchase agreements at thrifts.

In the above interest rate definitions, we have implicitly assumed that $\boldsymbol{m}_s$ contains three aggregated elements. In other words, there exists a vector of investment vehicles $\boldsymbol{a}_s$ that are weakly separable in the blocks described above.[11]

The size of our simulation is equal to 100, i.e. $T = 100$. We chose a small simulation size because of the explosive nature of the PE approach for large simulations when the solutions are not stationary processes.[12] This volatile behavior of the PE approach results from the size of the set containing the exogenous state variables becoming larger as T increases. The increased size of the exogenous state variable's set presents the PE algorithm with the difficult task of trying to determine which observations in the state space are relevant and which ones are not. Because each observation is weighted equally in the PE approach weights, as more and more irrelevant exogenous state observations occur the PE method becomes explosive trying to fit the irrelevant observations with the relevant states. To overcome this problem of endogenous oversampling for large simulations a number of small simulations can be produced.[13]

4.1 Limiting Case

Table 2 lists the summary statistics of the PE solutions to the limiting case, $\sigma \to 0$, $\alpha \to 0$. The initial values for $\boldsymbol{v}^0$ were set equal to zero except for the intercept terms. These coefficients were set equal to the natural logarithm of the expected value of $g_i(\mathbf{1}_{s+1}, \boldsymbol{\phi}_{s+1})$, where $i = 1, 2, 3, 4,$.[14] The starting values for $\{\boldsymbol{z}_s^0\}$ were also set equal to the expected values of $g_i(\mathbf{1}_{s+1}, \boldsymbol{\phi}_{s+1})$.[15]

With these initial values, the PE algorithm took 3494 interactions to converge. As shown in table 2, the DHM-stat for the solution rejects the hypothesis that the PE solutions equal

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[11] Barnett, Hinich and Yue (1991) used this same sub-nested blocking of a_s to obtain the GMM estimates for this model.

[12] Marcet and Marshall (1992) provide a set of conditions and an alternative PE approach that overcomes the problems associated with unstable solutions. They advocate taking a large number of samples of random disturbances to generate the exogenous state variables and use the sample average of the polynomial's coefficient estimates, v_i's, as the coefficients for approximating the expectation operators.

[13] See Marshall (1992) and Marcet and Marimon (1991) for examples of this approach.

[14] This is the method that is used by Bansal et. al. (1992).

[15] Another equally valid method for setting the initial values of v_i's is to regress $g_{i,s}$ on $\exp[poly(\tilde{\sigma}_s, v_i)]$ with $\{z_s\}$ equal to the steady state solution.

the 'true' solutions. This result goes against the findings of Den Haan and Marcet (1994). In simulation testing, Den Haan and Marcet found the PE algorithm with a first-order approximating function produced solutions with a DHM-stat higher than the critical 95% value too often and lower than the critical 5% value not often enough. Hence, our solution is at odds with the findings of Den Haan and Marcet. However, as we will see the PE approach produces solution to the optimization problem for other values of σ and α in which the DHM-stat agrees with the simulation results of Den Haan and Marcet.

Using the PE solution from the limiting case, we apply homotopy theory to find the solutions to the cases where $\sigma \to 0$, but α is allowed to move to 0.7, i.e. the convex combination of $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}^*$ is $\tau\, 0.7$.[16] When $\alpha = 0.1,\ 0.2,\ 0.3,\ 0.4$, the PE solutions and polynomial coefficient estimates from the model $\sigma \to 0$ and $\alpha \to 0$ were used as the initial starting values for the Newton-like solutions of the system of equations and as initial values for $\boldsymbol{v}^0$ and $\{\boldsymbol{z}_s^0\}$. Initially, as the value of α increased the number of iterations needed for convergence also grew. However, when $\alpha = 0.3$ the PE algorithm converged in 246 iterations, whereas it took the PE algorithm nearly 3,000 iterations to converge when $\alpha = 0.2$. Raising α to 0.40, increased the number of iteration for the PE method to 251.

For the two cases $\alpha = 0.5,\ 0.6$, the PE algorithm failed to converge within 40,000 iterations when the limiting case's solutions were used as initial values. We chose to redefine the homotopy function by respectively using the PE solutions and their polynomial coefficients from the models $\alpha = 0.45,\ 0.5$ as the initial starting values for the PE algorithm. We first tried to use the PE solutions from $\alpha = 0.4$ as the starting values for the case $\alpha = 0.5$, but the PE algorithm failed to converge. We then used these same starting values for the case $\alpha = 0.45$, and the PE method converged. Lastly, the starting values for the case $\alpha = 0.7$ are the PE solutions from the model, $\alpha = 0.6$.

As seen in table 4 and 5 and in the fact that the PE method failed to converge in some cases, using the PE solutions from a model close to the unsolved one lowers the number of iterations involved in the PE approach and enables an otherwise explosive algorithm to be converge.

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[16] By keeping σ constant and allowing α to change, the level of relative risk aversion is held constant at one, but the theoretical monetary aggregate changes from a Cobb-Douglas function to a CES aggregate function. Furthermore, this change in parameter values moves us closer to the empirically estimated value of α, which Barnett, Hinich and Yue (1991) found to be 0.8426. Unfortunately, we were unable to obtain convergence with the PE approach for values of α larger than 0.7.

4.2 Higher Risk Case

In this subsection we present the PE solutions for our dynamic optimization problem when the value of σ becomes smaller, i.e. the level of relative risk aversion increases.[17] Unlike the PE solutions in the previous subsection we use the convergence criterion advocated by Marshall (1992) to determine the stopping point for the PE algorithm. We initially tried the convergence criterion in Eq. (13) but the PE algorithm failed to converge within 40,000 iterations.

Marshall argues that the PE algorithm converges to an approximate equilibrium if the coefficient vector, $\boldsymbol{v}$, is the optimal prediction vector. The $\boldsymbol{v}_i$'s are considered optimal coefficient vectors if they minimize the sum of squared residuals associated with the PE approaches' least square regressions. Hence, the convergence criterion used in this subsection is

$$\max_i \left| \frac{SSR(\boldsymbol{v}_i^k) - SSR(\boldsymbol{v}_i^{k-1})}{SSR(\boldsymbol{v}_i^{k-1})} \right| \leq \xi \tag{20}$$

where $SSR(\boldsymbol{v}_i^k) = \sum_{s=t+1}^{t+T} \nu_{isk}^2$, and ν_{isk} is the s observation of the residuals associated with the estimated coefficient vector from the kth iteration. As before all the results were calculated using $\xi = 0.03$.

The summary statistics for the PE solutions are found in table 5 and 6. In each case we used homotopy theory to provide the initial values for the polynomial's coefficients. For those models with $\sigma = -0.1, -0.2, -0.3$ we used the parameter values, $\boldsymbol{v}^*$, and PE solutions, $\{\boldsymbol{z}_s\}$, from the limiting case as initial values, whereas for the cases $\sigma = -0.4, -0.5, -0.6, -0.7$ we used the PE's solutions from the model with $\sigma = -0.3$ and $\alpha \rightarrow 0.0$.

The number of iteration to convergence is noticeably smaller in table 5 and 6 than those found table 3 and 4. This can be attributed to the different convergence criterion. However, what is important in table 5 and 6 is how homotopy theory reduces the number of iterations required for the PE algorithm to converge. In table 5, notice how the number of iterations increases from 16, when $\sigma = -0.1$, to 23 for $\sigma = -0.3$. But when the homotopy function is

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[17] The reader will note that as σ becomes negative the Euler equations (5)-(8) include both current and future periods theoretical monetary aggregate. This, along with the complex nature of the implicit system of equations (5)-(8), creates a serious problem for the PE approach. We can only hypothesis that this is the case, but in the future we desire to determine if this is correct by calculating an approximate solution to our dynamic optimization problem with one of the other Euler equation techniques [Coleman (1990, 1991), Baxter (1991), or Baxter, Crucini, and Rouwenhorst (1990)].

redefined for the solutions at $\sigma = -0.3$, the number of iterations drops to 14 for $\sigma = -0.4$ and grows to 17 as σ increases to 0.7.

5 Conclusion

Homotopy theory can aid a numerical algorithm in finding a solution. This is especially helpful when trying to solve a nonlinear rational expectation problem since many of the existing solution methods are numerical. In this paper we have shown how homotopies can be used in conjunction with the Parameterized Expectation Approach to provide the initial starting values for both the PE algorithm and the Newton-like algorithm used to solve a nonlinear system of equation at each iteration of the PE method. By applying the PE approach to a representative agent's optimization problem of maximizing a time separable utility function over a aggregate consumption good and a aggregate monetary good, we found that the PE algorithm is more likely to converge with a lower number of iterations when homotopy theory is used.

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ρ	0.89750
β	0.95350
δ_1	0.46560
δ_2	0.33710
δ_3	0.19730
a_{11}	0.00252
a_{12}	0.00040
a_{13}	0.00178
a_{21}	0.00149
a_{22}	0.00414
$A_{1_{11}}$	0.90260
$A_{1_{21}}$	-0.00031
$A_{1_{31}}$	-0.01017
$A_{1_{12}}$	0.00755
$A_{1_{22}}$	0.99056
$A_{1_{32}}$	0.01083
$A_{1_{13}}$	0.09682
$A_{1_{23}}$	0.00181
$A_{1_{33}}$	0.98477
$A_{2_{11}}$	0.65470
$A_{2_{21}}$	-0.32513
$A_{2_{12}}$	-0.02609
$A_{2_{22}}$	-0.15840
$H_{1_{11}}$	0.00817
$H_{1_{12}}$	0.00181
$H_{1_{22}}$	0.00192
$H_{1_{13}}$	0.00451
$H_{1_{23}}$	0.00148
$H_{1_{33}}$	0.00352
$H_{2_{11}}$	0.00246
$H_{2_{12}}$	2.0D-05
$H_{2_{22}}$	0.00624

Table 1: Parameter Values

$\sigma \to 0$ $\alpha \to 0$	
	μ
X	24655.300
m_1	6954.800
m_2	10662.800
m_3	17586.900
DHM-stat	0.0018 (1.0)
Iter.	3494

Table 2: Limiting Case

Case		μ
$\sigma \to 0.0$ $\alpha = 0.1$	X	24061.4D+3
	m_1	65120.4D+2
	m_2	10310.5D+3
	m_3	18445.3D+3
	DHM-stat	98.99 (0.0)
	Iter.	1028
$\sigma \to 0.0$ $\alpha = 0.2$	X	66525D+8
	m_1	17171D+8
	m_2	28807D+8
	m_3	55460D+8
	DHM-stat	98.99 (0.0)
	Iter.	2904
$\sigma \to 0.0$ $\alpha = 0.3$	X	12255.5D+1
	m_1	29662
	m_2	53634.3
	m_3	11358.5D+1
	DHM-stat	98.99 (0.0)
	Iter.	246
$\sigma \to 0.0$ $\alpha = 0.4$	X	12545D+1
	m_1	27670
	m_2	55341.8
	m_3	13296.6D+1
	DHM-stat	98.99 (0.0)
	Iter.	251

Table 3: Summary stat., limiting case solution as initial values

Case		μ
$\sigma \rightarrow 0.0$ $\alpha = 0.5$	X	30765.5D+1
	m_1	58800.4
	m_2	13595.3D+1
	m_3	38672.3D+1
	DHM-stat	98.99 (0.0)
	Iter.	134
$\sigma \rightarrow 0.0$ $\alpha = 0.6$	X	15495D+3
	m_1	23208.6D+2
	m_2	67198D+2
	m_3	24117.2D+2
	DHM-stat	98.99 (0.0)
	Iter.	593
$\sigma \rightarrow 0.0$ $\alpha = 0.7$	X	17837D+4
	m_1	16313.1D+3
	m_2	68115.6D+3
	m_3	38527D+4
	DHM-stat	98.99 (0.0)
	Iter.	397

Table 4: Summary stat., $\alpha = 0.45, 0.5, 0.6$ solutions respectively as initial values

Case		μ
$\sigma = -0.1$ $\alpha \to 0.0$	X	11486.50
	m_1	3289.34
	m_2	5211.88
	m_3	7938.32
	DHM-stat	98.99 (0.0)
	Iter.	16
$\sigma = -0.2$ $\alpha \to 0.0$	X	7973.33
	m_1	2314.42
	m_2	3742.51
	m_3	5637.62
	DHM-stat	98.99 (0.0)
	Iter.	17
$\sigma = -0.3$ $\alpha \to 0.0$	X	11230.4
	m_1	3179.4
	m_2	4875.3
	m_3	7877.2
	DHM-stat	98.99 (0.0)
	Iter.	23

Table 5: Summary stat. limiting case as initial values

Case		μ
$\sigma = -0.4$ $\alpha \rightarrow 0.0$	X	6147.51
	m_1	1791.93
	m_2	2855.18
	m_3	4586.94
	DHM-stat	98.97 (0.0)
	Iter.	14
$\sigma = -0.5$ $\alpha \rightarrow 0.0$	X	3500.93
	m_1	1040.49
	m_2	1706.88
	m_3	2784.05
	DHM-stat	98.97 (0.0)
	Iter.	15
$\sigma = -0.6$ $\alpha \rightarrow 0.0$	X	2138.06
	m_1	644.63
	m_2	1081.45
	m_3	1780.79
	DHM-stat	98.97 (0.0)
	Iter.	15
$\sigma = -0.7$ $\alpha \rightarrow 0.0$	X	1412.10
	m_1	424.82
	m_2	710.43
	m_3	1201.31
	DHM-stat	98.99 (0.0)
	Iter.	17

Table 6: Summary stat., $\sigma = -0.3$, $\alpha \rightarrow 0$ solutions as initial values