PRODUCTIVITY AND INTERMEDIATE PRODUCTS:

A Frontier Approach

by

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Nishimizu and Hulten (1978) developed a model for measuring productivity growth when intermediate inputs are explicitly recognized. they show "...that the aggregate rate of productivity change is the weighted *sum* of the sectoral rates" (p. 353). To derive this result, the authors assume that the inputs are "...allocated efficiently among sectors" (p. 352). The weights used to sum the sectoral rates relies on prices, and thus it is implicitly assumed that the technology operates in an allocatively efficient manner, to use the terminology of Farrell (1957).

The purpose of this paper is to introduce a frontier model for productivity measurement that does not require that inputs are efficiently allocated among sectors or that prices are available. In particular we develop a network activity analysis model that explicitly recognizes that some inputs are produced and consumed within the production technology. Here we differ from Koopmans (1951) by assuming that the intermediate inputs may also be final output. This assumption is in line with current international trade theory, where intermediate inputs are tradable.

Our model consists of two production units that are interconnected in a network to form a production technology. The first node or production unit produces outputs, some of which are used as in puts in the second unit. These intermediate inputs may also be final outputs, in this case we consider these as "spare parts." Our network technology is formalized as an activity analysis model, with each coefficient being an observation. The frontier formed from the observation is therefore the best practice frontier. Frequently such models are called nonparametric or DEA (Data Envelopment Analysis), see Charnes, Cooper and Rhodes (1978).

The productivity measure employed here is the so-called Malmquist productivity index. This index consists of ratios of distance functions. Here these distance functions are defined on the network technology and they are computed using linear programming techniques.

1. The Frontier Production Technology

In order to construct the frontier production technologies relative to which productivity is measured, we assume that these are k = 1,...,K observations of exogenous inputs $x^k \in \mathbb{R}^N_+$ and final outputs $y^k \in \mathbb{R}^M_+$. Recall that some of the final outputs may be used as intermediate inputs. We identify these below. In this paper we assume that there are two production units or nodes that make up the technology. (This assumption can be generalized, but for now it will do). We denote the exogenous inputs that are used in process i, i = 1,2 by ${}^{i}x$, and clearly,

(1) $x^k \geq {}^1x + {}^2x.$

In the same manner, total output is the sum of outputs from node 1 and 2, noting that some of the outputs from 1 are used as inputs in node 2. The final output is thus

(2) $y^k = {}^f_1y + {}^f_2y,$

i.e., the sum of outputs from node 1 and 2. Recall that part of the production in node 1 will use as intermediate inputs. These are denoted by ${}^{2}_{1}y.$ Thus the total output from node 1 equals

(3} $\quad {}^{2}_{1}y + {}^{1}_{f}y.$

The above network technology is illustrated in Figure 1.

Figure 1: The network technology.

In the figure, the two production units or nodes are represented by circles. The exogenous inputs x^k are allocated between them with ${}^{i}x$ going to the ith note. The final outputs y^k are the vector sum of outputs ${}^{f}_{1}y$ from 1 and ${}^{f}_{2}y$ from 2. The intermediate input vector produced at 1 and used at 2 is denoted by ${}^{2}_{1}y.$ The network technology consists of the box and its interconnected nodes. In traditional economics, the technology consists of the pairs (x^k, y^k) such that x^k can product y^k, and no mentioning of nodes. Hence in such a model one cannot study the allocation of input or outputs among the nodes.

Following Shephard and Färe (1975) and Färe (1991), we may formulate the network technology as an activity analysis or DEA model. First consider node 2. This node can be written as the DEA model

$$(4)\quad \begin{aligned} &(a)\ \sum_{k=1}^{K} \lambda k\ {}^{f}_{2}y_{km} \geq {}^{f}_{2}y_{m},\ m = 1,\ldots,M,\\ &(b)\ \sum_{k=1}^{K} \lambda k\ {}^{2}_{1}y_{km} \leq {}^{2}_{1}y_{m},\ m = 1,\ldots,M,\\ &(c)\ \sum_{k=1}^{K} \lambda k\ {}^{2}y_{kn} \leq {}^{2}x_{n},\ n = 1,\ldots,N,\\ &(d)\ \lambda k \geq 0,\ k = 1,\ldots,K, \end{aligned}$$

where λ_k denotes the kith intensity variable. Here it is only required to be nonnegative, implying that node 2 exhibits constant returns to scale. Among the four sets of constraints, (b) is the one that models the consumption in 2 of the output from 1, i.e., it models the "use side" of the intermediate input vector ${}^{2}_{1}y$. Recall that we have assumed that there are k = 1,...,K observations of inputs and outputs, in particular of ${}^{f}_{2}y_{km}$, ${}^{2}_{1}y_{km}$ and ${}^{2}x_{kn}$.

As an activity analysis model node 1 is written as

$$(5)\quad \begin{aligned} &(e)\ \sum_{k=1}^{K} \mu_k \left({}^{2}_{1}y_{km} + {}^{f}_{1}y_{km}\right) \geq \left({}^{2}_{1}y_{m} + {}^{f}_{1}y_{m}\right),\ m = 1,\ldots,M,\\ &(f)\ \sum_{k=1}^{K} \mu_k\ {}^{1}x_{kn} \leq {}^{1}x_{n},\ n = 1,\ldots,N,\\ &(g)\ \mu_k \geq 0,\ k = 1,\ldots,K \end{aligned}$$

where μ_k, k = 1,...,K denote the intensity variables associated with the first node. Constraint (e) shows how much of node one's output goes to node 2 ${}^{2}_{1}y_{m}$ and how much becomes final output ${}^{f}_{1}y_{m}$.

The network technology consists of expressions (1), (2), (4) and (5). Since each node exhibits constant returns to scale, so does the network technology. In addition it satisfies strong disposability of inputs and outputs since the nodes do satisfy strong disposability of inputs and outputs.

2. The Malmquist Productivity Index in a Network Framework

As was mentioned above, the Malmquist productivity index consists of ratios of distance functions. Here we will assume that the reference technology satisfies constant returns to scale, in which case the input and output distance functions are reciprocal to each other (Färe, 1988). This assumption is necessary and sufficient for productivity in the one input one output case to equal the ratio of average products (Färe and Grosskopf (1994:a)).

Recall that the output distance function is defined on the technology S^t, t = 1,...,T at t by

(6) $$D_o^t(x^t,y^t) = \inf\{\Theta:(x^t,y^t/\Theta) \in S^t\},$$

where $S^t = \{(x^t, y^t): x^t \text{ can produce } y^t\}$.

Following Färe, Grosskopf, Lindgren and Roos (1989) the output oriented Malmquist productivity index is

(7) $$M_o(t,\ t+1) = \left[\frac{D_o^t(x^{t+1},y^{t+1})}{D_o^t(x^t,y^t)}\ \frac{D_o^{t+1}(x^{t+1},y^{t+1})}{D_o^{t+1}(x^t,y^t)}\right]^{1/2}.$$

This index is the geometric mean of two indexes as defined by Caves, Christensen and Diewert (1982) namely

(8) $$M_{CCD}^t = \frac{D_o^t(x^{t+1},y^{t+1})}{D_o^t(x^t,y^t)}$$

and

(9) $$M_{CCD}^{t+1} = \frac{D_o^{t+1}(x^{t+1},y^{t+1})}{D_o^{t+1}(x^t,y^t)}.$$

Moreover, Färe, Grosskopf, Lindgren and Roos (1989) showed that the Malmquist index (7) can be decomposed into two components, one measuring efficiency change and another measuring technical change. These components are

(10) $$ECH = \frac{D_o^{t+1}(x^{t+1},y^{t+1})}{d_o^t(x^t,y^t)}$$

(11) $$TCH = \left(\frac{D_o^t(x^{t+1}, y^{t+1})}{D_o^{t+1}(x^{t+1}, y^{t+1})} \frac{D_o^t(x^t, y^t)}{D_o^{t+1}(x^t, y^t)} \right)^{1/2}$$

respectively. For additional decompositions of the efficiency change component, see Färe, Grosskopf, and Lovell (1984), and for a decomposition of TCH, see Färe and Grosskopf (1994:b).

3. Computation of Productivity

In this section we demonstrate how the Malmquist productivity change can be computed. We combine the ideas of sections 1 and 2 to formulate a linear programming problem whose solutions yield the appropriate value of the required distance function. First though we recognize that our k = 1,...,K observations should form a panel in the sense that data for t = 1,...,T are available. In this case we can calculate the distance function $D_o^t(x^{k',t}, y^{k',t})$ for $k' = 1,\ldots,K$ as

(12)
$$\left(D_o^t(x^{k',t}, y^{k',t})\right)^{-1} = \max \Theta$$
$$\text{s.t.} \qquad \Theta\, y^{k',t} = {}_1^f y + {}_2^f y$$
$$\sum_{k=1}^{K} \lambda_k \, {}_2^f y_{km}^t \geq {}_2^f y_m, \; m = 1,\ldots,M,$$
$$\sum_{k=1}^{K} \lambda_k \, {}_1^2 y_{km}^t \leq {}_1^2 y_m, \; m = 1,\ldots,M,$$
$$\lambda_k \geq 0, \; k = 1,\ldots,K$$
$$\sum_{k=1}^{K} \mu_k \left({}_1^2 y_{km}^t + {}_1^f y_{km}^t \right) \geq \left({}_1^2 y_m + {}_1^f y_m \right), \; m = 1,\ldots,M,$$
$$\sum_{k=1}^{K} \mu_k \, {}^1 x_{kn}^t \leq {}^1 x_n, \; n = 1,\ldots,N,$$
$$\mu_n \geq 0, \; k = 1,\ldots,K$$
$${}^1 x_n + {}^2 x_n \leq x_{k'n}^t \quad n = 1,\ldots,N.$$

The mixed period values of the distance function say $D_o^{t+1}(x^{k',t}, y^{k',t})$ are computed like (12). However one must note that the reference technology in this case consists of data from period (t + 1).

4. Summary

In this paper we have provided an example of a simple technology which explicitly accounts for intermediate inputs. We have also shown how such a technology could be reported in an activity analysis framework as a "network." This in turn implies that these models could be used in "DEA" analysis, to model firms for which information on intermediate production is available. This would provide a "better"[1] representation of technology than the usual "black box" input and final output models. This model would also provide, as part of the solution to the "efficiency" problem, optimal allocations of inputs to the various nodes, including intermediate production.

Since the resulting "DEA" problem with intermediate products is essentially a distance function, one may also use these functions to construct Malmquist productivity indexes, which was discussed in sections 2 and 3. This has several advantages over traditional productivity approaches. It identifies the frontier of technology, and it does not in principle require information on prices of inputs and outputs. This approach also allows for and identifies inefficient allocations, including inefficiency due to misallocation of inputs among nodes.

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[1]Better in the sense of providing a "tighter fit" to the data. See Färe and Whittaker (199).

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