A "One-line" Simulator for Maxima or Minima on Drifting Brownian Paths

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Abstract

A simple transform of a standard uniform variate is given for simulation of the maximum attained by a Wiener process with drift, conditioned upon the level attained by the process over an arbitrary time interval. The transform arises directly from inversion of the joint distribution function of the maximum and the final Wiener process level.

1 Introduction

Generating random observations that possess a certain probability distribution is often accomplished by inverting the cumulative probability function. In the simplest cases, unit uniform pseudo-random numbers are functionally transformed to numbers that are exactly distributed according to a probability law being studied. Perhaps the best known example of this strategy is the Box and Muller method for generating pseudo-random normally distributed numbers, found in [2].

This note presents a simple transformation, which maps a uniformly distributed random number to a number which is exactly distributed as the maxima (or minima) of paths of a Wiener process with arbitrary diffusion and drift, conditioned on a specified level of the process at the end of a time interval. The transform entails one simple expression, taking as its arguments a unit uniform pseudo-random number, and the ending level of the path.

While the problem of constructing Brownian paths with a *given* extreme has been treated in [1], there appears to be no advancement of simple methods to simulate Gaussian maxima with respect to paths conditioned on the ending process level. This note offers such a method, and, by extension, a alternative, direct, method for unconditional simulation of extremes as well.

A direct method offers several advantages over alternative techniques, such as suggested in [4]. Direct transformation is more parsimonious, because the alternatives implicitly rely on asymptotic convergence to continuous time distributions. Also, samples generated by transformation will not be subject to numerical problems, such as those observed in [3]. Further, the "conditional" aspect of this transform facilitates its use in applications that require control of final outcomes. An example of effective application of the conditional feature of the transformation is in the Monte Carlo design for studies of complicated financial trading or hedging procedures. These studies are often concerned with distributions of outcomes under alternative future speculative market scenarios. Use of this method to provide properly distributed values for the extreme level with respect to the condition of the final level, coupled with the method in [1] to construct dually-conditioned Browinian paths, could facilitate applications which require control of both path functionals.

2 Methodology

The joint probability of the maximum and the terminal level of a Wiener process with drift is presented in [5][1]. Let T denote an elapsed time from inception of the process. Let Y denote the maximum over that interval, and x_T denote the level attained by a Wiener process at T. If the Wiener process has drift c and diffusion σ^2, then the probability law of the maximum and final level, say, $F(x_T, Y; T, c, \sigma)$ is:

$$dF(x_T, Y; T, c, \sigma) = \frac{2(2Y - x_T)}{\sqrt{(2\pi\ \sigma^4 T^3)}}\ \exp\left(-\frac{(2Y - x_t)^2}{2\sigma^2 T} + \frac{c\ x_T}{\sigma^2} - \frac{c^2\ T}{2\ \sigma^2}\right) dY\ dx_T \quad (1)$$

[1]The joint probability of the maximum, terminal level, and the time of the maximum is provided in that article; here, only the marginal distribution of the first two is required.

The process' finite dimensional probability distribution is of course the normal distribution, say, $G(x_T; T, c, \sigma)$:

$$dG(x_T; T, c, \sigma) = \frac{1}{\sqrt{(2\pi\ \sigma^2 T)}} \exp\left(-\frac{(x_t\ -\ cT)^2}{2\sigma^2 T}\right) dx_T \tag{2}$$

Let $C(Y|\xi)$ denote the probability law of Y conditioned on $x_T = \xi$. It is elementary to obtain the density, dC, as the ratio of the joint marginal density (1) and the conditioning variate's density (2). After simplification, this density is:

$$dC(Y \mid \xi; T, \sigma) = \frac{2\ (2Y - \xi)}{\sigma^2 T} \exp\left(-2Y\ \frac{(Y - \xi)}{\sigma^2 T}\right) dY \tag{3}$$

The conditional density is independent of the drift parameter, c. Accordingly, the inversion of the cumulative distribution function of the conditioned maximum will be independent of the process' drift. Moreover, the form is that of a simple exponential, and so has an elementary analytic integral. Let $Q(M \mid \xi)$ denote the (complementary) cumulative probability, i.e.:

$$\begin{aligned} Q(M \mid \xi) &\equiv Pr(Y \geq M | x_T = \xi) \\ &= \int_M^\infty \frac{2\ (2\psi - \xi)}{\sigma^2 T} \exp\left(-2\psi\ \frac{(\psi - \xi)}{\sigma^2 T}\right) d\psi \\ &= \exp\left(-2\ M\ \frac{(M - \xi)}{\sigma^2 T}\right) \end{aligned} \tag{4}$$

This probability function is trivially invertible. Let U_M denote the negative of the log of $Q(M, xT)$, whence:

$$U_M = -\ \ln\left(Q(M \mid \xi)\right) = 2\ M\ \frac{(M - \xi)}{\sigma^2 T}.$$

Since M must not be less than ξ, then:

$$M = \frac{1}{2}\xi\ +\ \frac{1}{2}\left(\xi^2\ +\ 2U_M\ \sigma^2 T\ \right)^{1/2}\ ,\ M > \xi,\ \ 0 < U_M < 1. \tag{5}$$

If U_M is simulated by generating a uniform pseudo-random number, say, u, then (5) will produce another pseudo-random number, say, ψ, distributed as the maximum conditioned on the final level attained by the Wiener process, as in (3).

3 Comments

The "one-line" simulation engine provided by (5) could find application in, for example, a problem of the following type. Say that some sequential procedure were modelled with underlying causation following a Wiener process, but the procedure were itself too complicated for its results to admit a closed form solution for either its stochastic differential equation or for its finite-dimensional distributions. Further, say it were known that a certain level of the underlying process, regardless of when that level were attained, would result in "ruin", but, in the alternative, the process would run to a horizon date. In a Monte Carlo study of the procedure's results, especially one which employed Brownian Bridge conditioning, application of (5) as part of the method would provide two advantages. First, if the result exceeded the "ruin level", that trial does not need a Monte Carlo to be generated. Second, recording the frequency and conditioning of the "hits" would provide a direct assessment of the probability of ruin, conditioned on final process levels.

Sequences of exactly distributed *unconditional* Wiener maxima can be attained with only a small increase in cost. Such application requires a pair of random numbers (u, x). The random number u is taken as before, and x is generated as a normal variate with mean $c\,T$ and variance $\sigma^2\,T$. Then the resultant random numbers will be distributed as the maxima of all Wiener paths over $(0, T)$ having those parameters.

The transform can also be used directly to generate Wiener process minima. The minimum level attained by a Wiener path over an interval, conditioned upon x_T, is distributed as the maximum, conditioned upon the negative of x_T.

Finally, it should be noted that this method *cannot* be employed to simulate a "range" for Wiener paths, *i.e*, the difference between the maximum and minimum levels. The reason for this is a well known, and intuitively obvious, result: the maximum and minimum levels of a Wiener path are not independent. It does not seem likely that a method to generate simulated ranges can be attained from the inversion of a distribution function, since the distribution of the Wiener range functional is known only in terms of convergent infinite series.

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